Ordinary (0-form) symmetries

Symmetry transf. in QFT

$$\langle U_g(\Sigma)\, \Phi^i(y) \rangle = R(g)^i{}_j \langle \Phi^j(y) \rangle$$

Since the sym. generators are CONSERVED / COMMUTE WITH HAMILTONIAN, $U_g(\Sigma)$ is "topological" (as we will see)

In Field Theory, if S is invariant under sym group G, then there exists a CONSERVED CURRENT $\partial_\mu j^\mu = 0$

j s.t. if we take local transf

$$S[\Phi^i + \epsilon(x) \underbrace{M^i{}_j}_{\text{generator}} \Phi^j] - S[\Phi^i] = -\int \epsilon(x)\, \partial_\mu j^\mu(x) \qquad (*)$$

there is a current associated with any gen.

In QFT $\rightsquigarrow$ WI

$$i \langle \partial_\mu j^\mu(x)\, \Phi^i(y) \rangle = \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y) \rangle \qquad (\circ)$$

Dim. $\langle \partial_\mu j^\mu(x)\, \Phi^i(y) \rangle = N \int \mathcal{D}\Phi\, \partial_\mu j^\mu(x)\, \Phi^i(y)\, e^{iS[\Phi]} =$

$$\overset{(*)}{=} -N \int \mathcal{D}\Phi\, \frac{\delta}{\delta \epsilon(x)} S[\Phi^k + \epsilon(x) M^k{}_j \Phi^j]\Big|_{\epsilon=0} \Phi^i(y)\, e^{iS[\Phi]} =$$

$$= -\frac{1}{i} \frac{\delta}{\delta \epsilon(x)} \Big(N \int \mathcal{D}\Phi\, \Phi^i(y)\, e^{iS[\underbrace{\Phi^k + \epsilon M^k{}_j \Phi^j}_{= \Phi'^k \to \Phi^k = \Phi'^k - \epsilon M^k{}_j \Phi'^j}]} \Big) \Big|_{\epsilon=0}$$

$$= i \frac{\delta}{\delta \epsilon(x)} N \int \mathcal{D}\Phi' \left(\Phi'^i(y) - \epsilon(y) M^i{}_j \Phi'^j(y) \right) e^{iS[\Phi']} \Big|_{\epsilon=0}$$

$$= -i\, \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y) \rangle \quad //$$

'tHooft ANOMALY:

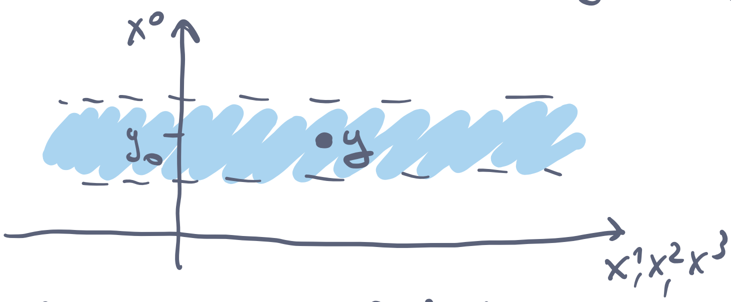
$\partial^{x_1}_\mu \langle j^\mu_A(x_1)\, j^\alpha(x_2) \rangle = \frac{i}{\pi} \epsilon^{\alpha\beta} \partial_\beta \delta(x_1 - x_2)$

$\int dx_1 \downarrow = 0$

We can now integrate the WI (o) and obtain

$$i \langle [Q, \Phi^i(y)]_{\text{eq.time}} \rangle = M^i{}_j \langle \Phi^j(y) \rangle \qquad \text{(canonical quantization)}$$

Dim. Integrate $i \langle \partial_\mu j^\mu(x)\, \Phi^i(y) \rangle = \delta^4(x-y) M^i{}_j \langle \Phi^j(y) \rangle$
over the domain $\Omega_\Sigma \equiv [y^0+\epsilon, y^0-\epsilon] \times \mathbb{R}^3$



$\int_{\mathbb{R}^3} \partial_i j^i = 0$

LHS: $\int_{\Omega_\Sigma} d^4x\, \partial_\mu j^\mu(x) = \int d^3x \left(j^0(y^0+\epsilon, \bar{x}) - j^0(y^0-\epsilon, \bar{x}) \right) =$

$= Q(y^0+\epsilon) - Q(y^0-\epsilon)$

$\langle (Q(y^0+\epsilon) - Q(y^0-\epsilon)) \Phi^i(y) \rangle = \langle 0 | T(Q(y^0+\epsilon) - Q(y^0-\epsilon)) \Phi^i(y) | 0 \rangle =$

$= \langle [\hat{Q}(y^0), \hat{\Phi}^i(y)] \rangle$

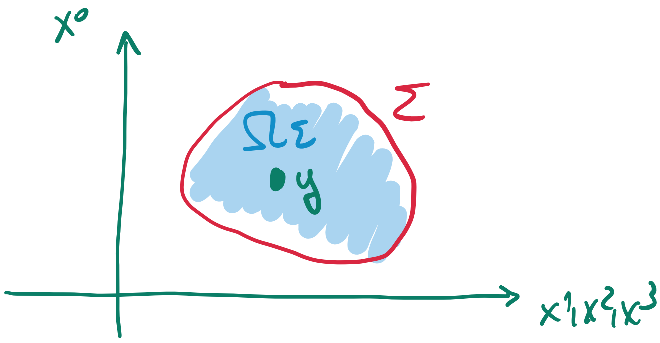
//

How does it work for extended objects?

?

Rewriting ordinary sym. transf.

$$i \langle Q(\Sigma)\, \Phi^i(y) \rangle = \mathrm{Link}(\Sigma, y)\, M^i{}_j \langle \Phi^j(y) \rangle$$



Charge Q on a time slice is generalized (Euclidean signature) to a charge $Q(\Sigma)$ on a 3d CLOSED subspace Σ

$$Q(\Sigma) \equiv \int_\Sigma * j$$

The commutation relations to LINK of Σ and y.
How do we derive this relation?

↓

Let's integrate w/ (•) on Ω_Σ

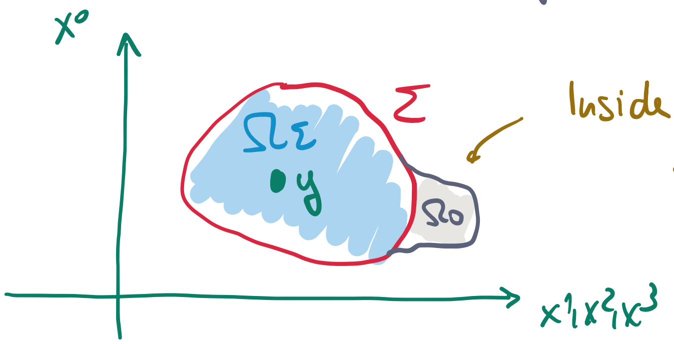
$$\text{LHS}: \int_{\Omega_\Sigma} \partial_\mu j^\mu d^4x = \int_{\Omega_\Sigma} d * j = \int_\Sigma * j = Q(\Sigma)$$

$$\hookrightarrow i\langle Q(\Sigma)\, \Phi^i(y) \rangle = \underbrace{\int_{\Omega_\Sigma} d^4x\, \delta^4(x-y)}_{\mathrm{Link}(\Sigma, y)}\, M^i{}_j \langle \Phi^j(y) \rangle$$

← TOPOLOGICAL INVARIANT

Also this is TOPOLOGICAL due to conserv. law:

under a contin. deform. $\Sigma \to \Sigma' = \Sigma + \partial\Omega_0 \quad y \in \Omega$



Inside Ω_0 there is NO INSERTION of local operators

in correlators

$=0 \longleftrightarrow \partial_\mu j^\mu = 0$

$$Q(\Sigma') = Q(\Sigma) + \int_{\partial\Omega_0} * j = Q(\Sigma) + \int_{\Omega_0} d * j = Q(\Sigma)$$

By exponentiating infinitesimal generators:

$$\langle U_g(\Sigma)\ \Phi^i(y) \rangle = R(g)^i{}_j \langle \Phi^j(y) \rangle \qquad \text{(if LINKED)}$$

charged operator (0-dim $\rightsquigarrow$ 0-form sym.)

TOPOLOGICAL unitary operator depending on $g \in G$ & Σ

$$\left[\frac{d}{d\alpha} U_{e^{i\alpha}}(\Sigma) \Big|_{\alpha=0} = i\, Q(\Sigma) \right]$$

[The insertion of the top. op. $U_g(\Sigma)$ can be removed at the cost of transforming all the local operators inside Σ. Equivalenty, we can say that if we deform the support passing through one loc. op. position, we act on it by g.]

Discrete symmetries

- $g \in G$ discrete

- $\langle U_g\, \Phi^i(y)\, U_g^{-1} \rangle = R(g)^i{}_j \langle \Phi^j(y) \rangle$

  related to a TOPOL. OPERATOR $U_g(\Sigma)$ s.t.

  $$\langle U_g(\Sigma)\, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \qquad \text{(if linked)}$$

- $U_g(\Sigma) U_{g'}(\Sigma) = U_{gg'}(\Sigma)$

Summary ORDINARY SYMMETRIES

$g \in G \longleftrightarrow$ TOPOL. OP. $U_g(\Sigma)$ $\qquad$ G cont/disc.

with

$$\langle U_g(\Sigma)\, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \qquad \text{(if linked)} \qquad (*)$$

topological: Σ is $(d-1-\underline{\underline{0}})$-obj. $\qquad$ not necessary topol.: $\Phi^i(y)$ is $\underline{\underline{0}}$-dim $\quad \leadsto$ "0-form symmetry"

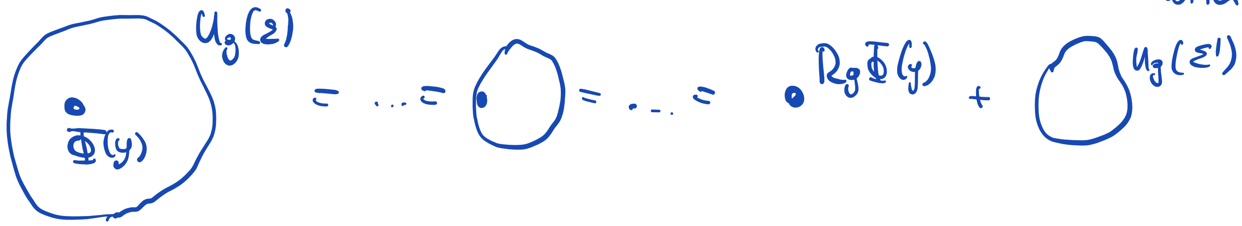
We have reduced the problem of finding symmetries to the problem of finding TOPOLOGICAL OPERATORS.

This can be generalized to topological $(d-1-p)$-op. i.e. p-form symmetries (we are now going to illustrate an example with $p=1$).

Observation: one can interpret $(*)$ pictorially:

$$\langle U_g(\Sigma)\Phi^i(y)\rangle = R^i{}_j(g)\langle\Phi^j(y)\rangle + 0 = R^i{}_j(g)\langle\Phi^j(y)\rangle + \langle U_g(\Sigma')\Phi^i(y)\rangle$$

with $\mathrm{Link}(\Sigma', y) =$



1-form symmetries in Maxwell theory

$$S[A] = -\frac{1}{2e^2}\int F\wedge *F \;=\; -\frac{1}{4e^2}\int d^4x\, F^{\mu\nu}F_{\mu\nu} \qquad (*)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\qquad A_\mu$ COMPACT $U(1)$ gauge field

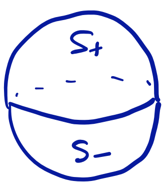
$\quad\Rightarrow$ ELECTRIC CHARGES are QUANTIZED

Gauge transformations:

$A \longrightarrow A + \lambda$ $\qquad$ with λ a (globally defined) closed 1-form (locally $\lambda = d\alpha$)

• $\alpha(x)$ is a $U(1)$ parameter, meaning that the well defined local transf. are $e^{i\alpha(x)} \rightarrow \alpha \sim \alpha + 2\pi$ $\quad (*)$

• let's take an S^2 and integrate F over it

S^2 

$$\int_{S^2} F = \int_{S^+} F + \int_{S^-} F = \int_{S^1} A^+ - \int_{S^1} A^- = \int_{S^1} \lambda =$$

$$= \alpha(2\pi) - \alpha(0) = 2\pi n \qquad n \in \mathbb{Z}$$

$U(1)$ gauge group $(*)$

$\Rightarrow$ For $U(1)$ gauge groups

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z} \quad \forall \Sigma_2 \qquad \& \qquad \int_\gamma \lambda \in 2\pi\mathbb{Z} \quad \forall \gamma$$

If $\int_\gamma \lambda = 0 \rightarrow \lambda = d\alpha$ e α è ben def. $\rightsquigarrow$ small gauge tranform.

$\Rightarrow$ For $U(1)$ gauge group, Wilson lines have integer charge $n \in \mathbb{Z}$:

$$W_n(\gamma) = e^{in\int_\gamma A} \quad \mapsto \quad e^{in\int_\gamma \lambda} W_n(\gamma)$$

S - duality

Let's rewrite the action as

$$S[F, \tilde{A}] = \frac{1}{2e^2}\int F \wedge * F + \frac{1}{2\pi}\int F \wedge d\tilde{A}$$

now indep. variable

new 1-form

Lagrange multiplier

- e.o.m. of $\tilde{A}$: $dF = 0 \longrightarrow$ Bianchi id.
- e.o.m of F : $\frac{1}{e^2} * F = \frac{1}{2\pi}\tilde{F}$ with $\tilde{F} = d\tilde{A}$

$\Rightarrow$ Integrating over $\tilde{A} \rightarrow S[A] = \frac{1}{2e^2}\int F \wedge * F$ with $F = dA$

Integrating over $F \longrightarrow S[\tilde{A}] = \frac{1}{2\tilde{e}^2}\int \tilde{F} \wedge * \tilde{F}$ with $\tilde{F} = d\tilde{A}$ and $\tilde{e}^2 = \frac{4\pi^2}{e^2}$

$\rightsquigarrow$ Example of DUALITY : the same theory has two equivalent presentations.

1-form symmetries

The e.o.m. of (★) are

$$\frac{1}{e^2}\partial_\mu F^{\mu\nu}=0 \quad \text{and} \quad \partial_\mu (\star F)^{\mu\nu}=0 \qquad \star F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$d\star F=0 \qquad\qquad\qquad dF=0$$

F and $\star F$ are two-forms that are closed

$\Rightarrow$ they define TWO 1-form symmetries with currents $J_e=\frac{1}{e^2}F$ and $J_m=\frac{1}{2\pi}\star F$

• The corresponding CONSERVED CHARGES are

- Electric flux

$$Q_e(\Sigma_2)=\frac{1}{e^2}\int_{\Sigma_2}\star F \sim \int_{\Sigma_2}\bar{E}\cdot d\bar{S} \leftrightarrow U(1)_e^{(1)}$$

- Magnetic flux

$$Q_m(\Sigma_2)=\frac{1}{2\pi}\int_{\Sigma_2}F \sim \int_{\Sigma_2}\bar{B}\cdot d\bar{S} \longleftrightarrow U(1)_m^{(1)}$$

• Under S-duality $J_e\longleftrightarrow J_m$

$$Q_e\longleftrightarrow Q_m$$

- Both $Q_e(S^2)$ & $Q_m(S^2)$ are TOPOLOGICAL under contin. deformations of Σ_2.
  $\Rightarrow$ there should be corresponding symmetries (whose related conserved quantities are the topol. op's)

time slice

- Sym. operators
$$U_e(\alpha_e, \Sigma_2) = e^{i\alpha_e Q_e(\Sigma_2)}$$
$$U_m(\alpha_m, \Sigma_2) = e^{i\alpha_m Q_m(\Sigma_2)}$$

with $\alpha_e \sim \alpha_e + 2\pi$ and $\alpha_m \sim \alpha_m + 2\pi$.

WILSON LINES

Crucial observation: shifting $A \to A + \xi$ with ξ a NON-QUANTIZED closed 1-form, still keeps the action invariant ($F \mapsto F$).

However, the Wilson lines are not invariant:

$$W_n(\gamma) \mapsto e^{i\beta} W_n(\gamma) \quad \text{with} \quad \beta = \int_\gamma \xi \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1)$$

$\Rightarrow$ this is not a redundancy, but a global symmetry of the type just described, i.e. a 1-form symmetry.

$\rightsquigarrow$ We want the associated W.I.

In order to derive the W.I., we use the usual trick: we change variable in P.I., applying a symmetry tranformation with non-const. parameter. For 1-form sym, this corresponds to ξ being NON-CLOSED:

$$\delta S = -\frac{1}{e^2}\int \xi \wedge d{*F} \qquad -\frac{1}{e^2}\int \underbrace{d\xi \wedge {*F}}_{d(\xi\wedge *F) + \xi \wedge d*F}$$

$$0 = \int e^{iS}\, i\delta S\, W + \int e^{iS}\delta W = \int e^{iS}\Big(-\frac{in}{e^2}\int \xi\wedge d{*F} + in\Big(\int_\gamma \xi\Big) W \qquad \int_\gamma \xi = \int \xi \wedge \delta^{d-1}(\gamma)$$

$\underbrace{}$ (d-1)-form s.t. $\int \eta \wedge \delta^{d-1}(\gamma) = \int_\gamma \eta \quad \forall \eta$

$$\Rightarrow \Big\langle \frac{1}{e^2} d{*F}\, W_n(\gamma)\Big\rangle = n\,\delta^{d-1}(\gamma)\langle W_n(\gamma)\rangle \qquad (*)$$

Consider a topologically trivial submanif. Σ_{d-2}.
It is the boundary of a D_{d-1}.

$$\int_{D_{d-1}} \delta^{d-1}(\gamma) = \gamma \cdot D_{d-1} = \mathrm{Link}(\gamma, \Sigma_{d-2})$$

Let's integrate $(*)$ over D_{d-1}, remembering $J_e = \frac{1}{e^2}F$:

$$\langle \underset{\frac{1}{e^2}\int_{\Sigma_{d-2}} *F}{\underset{\parallel}{Q_e(\Sigma_{d-2})}}\, W_n(\gamma)\rangle = n\, \mathrm{Link}(\Sigma_{d-2},\gamma)\langle W_n(\gamma)\rangle$$

$\Downarrow$

This shows what is the top. op. $U_e(\Sigma_{d-2})$ and that the Wilson lines are charged under this symmetry.

→ Symmetry transformation

In $d=4$

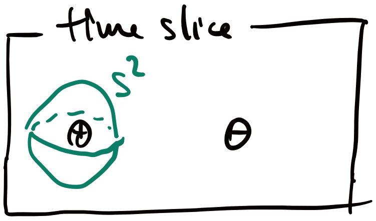
$$\langle U_{e^{i\alpha_E}}(\Sigma_2)\, e^{in\int_\gamma A}\rangle = e^{i\alpha_E n\, \mathrm{Link}(\Sigma_2,\gamma)} \langle e^{iq_E\int_\gamma A}\rangle$$

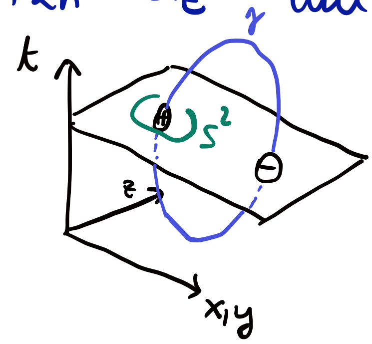
q_E (electr. charge of Wils. lin.)

$$U_e(\alpha_E,\Sigma_2) = e^{i\alpha_E Q_E(\Sigma_2)}$$

Sym. group is $U(1)$ $\alpha_E + 2\pi \sim \alpha_E$, due to quant. of $q_E = n$.

$\Sigma_2 = S^2$:





Summary:

- Sym op: $U_{e^{i\alpha_E}}(S^2) = e^{i\alpha_E Q_E(S^2)}$ 2d topol. op.

- Charged op: $e^{iq_E\int_\gamma A}$ $q_E = n \in \mathbb{Z}$

- Sym. group: $e^{i\alpha_E} \in U(1)$

↓

"ELECTRIC 1-form SYMMETRY"

't HOOFT LOOP $\quad T_n(\gamma)$

- Probe magn. part. (monopole)
- Closed line $\Leftrightarrow$ gauge invariance of dual photon
- $q_M = n \in \mathbb{Z}$
- obtain same formal expression as before when we dualize electric $\longleftrightarrow$ magnetic.

$\downarrow$

"MAGNETIC 1-form SYMMETRY"

Alternatively: inserting $T_n(\gamma_{d-3})$ in a correlator modifies P.I. domain, specifying b.c.

$$\int_{S^2} \frac{F}{2\pi} = n \quad \text{with} \quad \mathrm{Link}(S^2, \gamma_{d-3}) = 1$$

$$\langle \underset{\substack{\parallel \\ e^{i\alpha_m \int_{S^1} \frac{F}{2\pi}}}}{U_m(\alpha_m, \Sigma_2)}\, T_n(\gamma_{d-3}) \rangle \underset{\uparrow}{=} e^{in\alpha_m \mathrm{Link}(S^1, \gamma_{d-3})} \langle T_n(\gamma_{d-3}) \rangle$$

now $\int \frac{F}{2\pi}$ is fixed to a specific number $\Rightarrow$ it goes outside P.I.

Generalisations

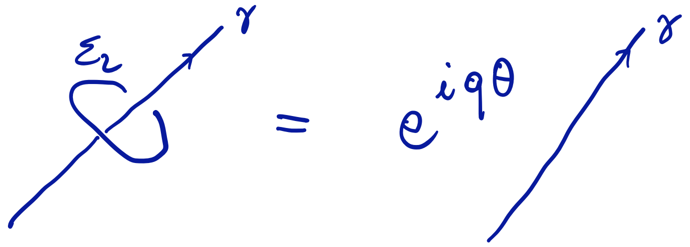
G p-form symmetry in d dim :

- Sym op. $U_g(\Sigma_{d-p-1})$
- Charged objects $W(q,\gamma_p)$
- Sym. transf. $\langle U_g(\Sigma_{d-p-1}) W(q,\gamma_p)\rangle = R(g)^q \langle W(q,\gamma_p)\rangle$ if linked

Take-home message : Existence of sym = Existence of TOPOLOGICAL OPERATORS

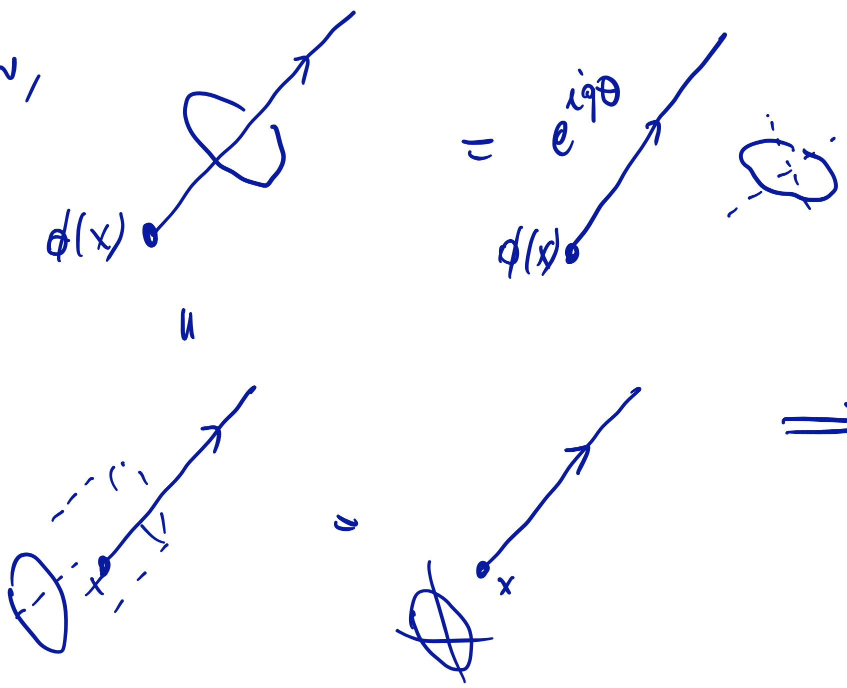
Adding matter

Let's remember how 1-form sym work:



If we now add charged fields $\phi(x)$ with charge q there will be <u>gauge invariant</u> lines that can end on the location x of the charged operator: in fact the gauge transf. on the extremum x of the WL is compensated by the gauge transf. of $\phi(x)$

Now,



$\Longrightarrow$ To be consistent:

$$e^{iq\theta} = 1$$

i.e. $\theta = \frac{2\pi k}{q}$

$$U(1)^{(1)} \to \mathbb{Z}_q^{(1)}$$

This is called "SCREENING" by lower dim. operators.
(Now line is uncharged.)

NON-ABELIAN GAUGE THEORIES

$$S = -\frac{1}{2g^2}\int \mathrm{Tr}(\mathbb{F} \wedge * \mathbb{F})$$

Gauge group G.

P.I. integrates over all G-bundles & their connections modulo gauge transformations.

G-bundle:

- cover $\{U_i\}$
- transition functs $g_{ij} : U_i \cap U_j \to G$ s.t. $g_{ij}\, g_{jk}\, g_{ki} = \mathbb{1}$

Connection A:

- local 1-forms $A_i \in \Omega^1(U_i, \mathfrak{g})$ glued as

$$A_j = g_{ij}^{-1} A_i\, g_{ij} + i g_{ij}^{-1} d g_{ij}$$

This should not be confused with gauge transf.; these are defined as $U_i : U_i \to G$ that acts as

$$A_i \mapsto U_i A_i U_i^{-1} + i U_i\, dU_i^{-1} \qquad g_{ij} \mapsto U_i\, g_{ij}\, U_j^{-1}$$

$$A_j = g_{ij}^{-1} A_i g_{ij} + i g_{ij}^{-1} d g_{ij}$$

$$A_i \mapsto U_i A_i U_i^{-1} + i U_i dU_i^{-1} \qquad g_{ij} \mapsto U_i g_{ij} U_j^{-1} = g'_{ij} \qquad g_{ij}^{-1} \mapsto U_j g_{ij}^{-1} U_i^{-1}$$

$$A'_j = U_j A_j U_j^{-1} + i U_j dU_j^{-1} =$$

$$= U_j g_{ij}^{-1} A_i g_{ij} U_j^{-1} + i U_j g_{ij}^{-1} dg_{ij} U_j^{-1} + i U_j dU_j^{-1}$$

$$= \underbrace{U_j g_{ij}^{-1} U_i^{-1}}\, \underbrace{U_i A_i U_i^{-1}}\, \underbrace{U_i g_{ij} U_j^{-1}} + i U_j g_{ij}^{-1} U_i^{-1} U_i dg_{ij} U_j^{-1} + i U_j dU_j^{-1}$$

$$g'^{-1}_{ij} \left(A'_i - i U_i dU_i^{-1}\right) g'_{ij} + i g'^{-1}_{ij} \left(dg'_{ij} - dU_i g_{ij} U_j^{-1} - U_i g_{ij} U_j^{-1} U_j dU_j^{-1} \right)$$

$$= g'^{-1}_{ij} A'_i g'_{ij} + i g'^{-1}_{ij} dg'_{ij} - i g'^{-1}_{ij} \cancel{U_i} dU_i^{-1} g'_{ij}$$

$$- i g'_{ij} \cancel{dU_i} U_i^{-1} g'_{ij} - i \cancel{U_j dU_j^{-1}} + \cancel{i U_j dU_j^{-1}}$$

$$U dU^{-1} + dU U^{-1} = \\ = d(UU^{-1}) = 0$$

Trasformazioni di gauge sono consistenti con funz. trans.:

$$\begin{tikzcd} A_i \arrow[r, "U_i"] \arrow[d, "g_{ij}"] & A'_i \arrow[d, "g'_{ij}"] \\ A_j \arrow[r, "U_j"] & A'_j \end{tikzcd}$$

Wilson lines.

Let us consider the Wilson lines

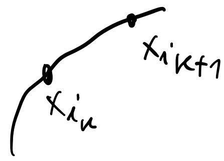
$$W_R(\gamma) = \mathrm{tr}_R\, P e^{i\int_\gamma A}$$

- To define the path ord. exp on an arbitrary curve γ we cut γ in small arcs $\gamma_i \subset U_i$ and

  use A_i to compute $\mathrm{hol}_{\gamma_i}(A_i) = P e^{i\int_{\gamma_i} A_i}$.

- Under gauge transf. $\mathrm{hol}_{\gamma_i} \to U_i(x_{in})\, \mathrm{hol}_{\gamma_i}\, U_i(x_{fin})^{-1}$

- $P e^{i\int_\gamma A} = \prod_{k=1,\dots} \mathrm{hol}_{\gamma_{i_k}}(A_{i_k})\, g_{i_k i_{k+1}}(x_{i_{k+1}})$



  and it transforms to $U_{i_{in}}(x_{IN})\, P e^{i\int_\gamma A}\, U_{i_{fin}}(x_{FIN})^{-1}$.

1-form symmetries.

Let us consider abelian case first:

- 1-form sym. $A \mapsto A + \lambda$ $\qquad$ λ closed 1-form

- On patch U_i : $\lambda|_{U_i} = d\eta_i$
- We can make a local gauge transform. $U_i = e^{-i\eta_i}$

  that corresponds to modifying the transition functs as

(*) $g_{ij} \mapsto e^{-i\eta_i} g_{ij}\, e^{i\eta_j} = g_{ij}\, t_{ij}$ $\qquad$ $t_{ij} = e^{-i(\eta_i - \eta_j)}$

Keeping A invariant.

$\to e^{i\oint_\gamma A} \mapsto e^{i\oint_\gamma A}\left(e^{-i\eta_{i_{in}}(x_{in})}\; e^{i\eta_{i_{fin}}(x_{fin})} = e^{i\int_\gamma \lambda}\right)$

In abelian theory the action of 1-form sym is (*).
This can be easily generalised to non-abelian gauge theories

- 1-form sym: $g_{ij} \mapsto g_{ij} t_{ij}$
- To preserve the cocycle condition t_{ij} must commute with any possible g_{ij} $\Rightarrow$ $t_{ij} \in Z(G)^{(*)}$.
  Moreover, it must happen that $t_{ij} t_{jk} t_{ki} = \mathbb{1}$
  [(*) $Z(G) = \{ z \in G \mid z g z^{-1} = g \quad \forall g \in G \}$ is the CENTER of the group G. It is ABELIAN.]
- Consider the Wilson line $W_R(\gamma) = \mathrm{tr}_R \, P e^{i \int_\gamma A}$ with R an irrep. Then $t_R \in Z(G)$ is a matrix proportional to the identity with prop. factor being the phase $\phi_R(t) = \frac{\mathrm{Tr}_R(t)}{\mathrm{Tr}_R(\mathbb{1})}$

The Wilson line then transforms as

$$W_R(\gamma) \mapsto \phi_R(g) \, W_R(\gamma)$$

$$\text{with} \quad g = \prod_{\substack{U_{ij} \text{ t.c.} \\ U_{ij} \cap \gamma \neq \emptyset}} t_{ij} \quad \in Z(G)$$

$\Rightarrow$ $W_R(\gamma)$ are line op.s charged under the 1-form sym $Z(G)^{(1)}$.

- For $G = SU(N)$ $\quad Z(G) = \mathbb{Z}_N$ and

$$W_R(\gamma) \mapsto e^{2\pi i q \frac{a}{N}} W_R(\gamma)$$

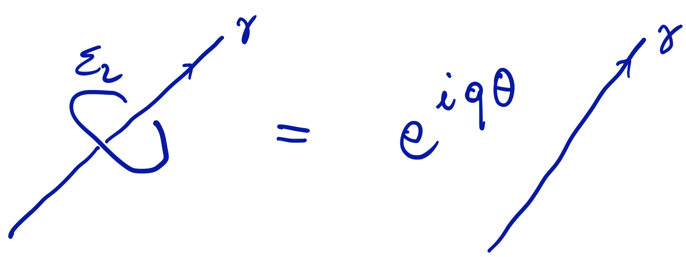
with $a = 0, 1, \dots, N-1$ labels $\mathbb{Z}_N$ group elem. and
$q = 0, \dots, N-1$ is the N-ality of the rep.

- There exist a SURFACE OPERATOR
$U(\Sigma_2)$ that will be the generator of a
$\mathbb{Z}_N$ one-form symmetry; in fact

$\langle U(\Sigma_2)\, W_N(\gamma) \dots \rangle$ and $\langle W_N(\gamma) \dots \rangle$ turn out to
differ by a factor $e^{2\pi i /N \cdot \mathrm{Link}(\Sigma_2, \gamma)}$

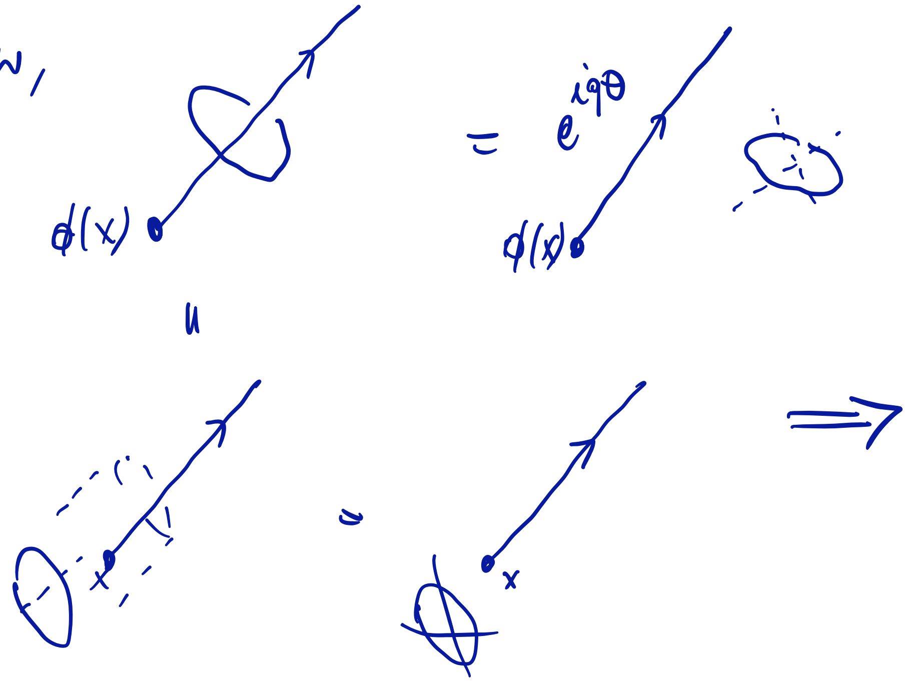
- Why $\mathbb{Z}_N$ instead of a continuous sym. like in
abelian gauge theories?

Let's remember how 1-form sym works:



If we now add charged fields $\phi(x)$ with charge q
there will be gauge invariant lines that can end
on the location x of the charged operator: in fact
the gauge transf. on the extremum x of the WL is
compensated by the gauge transf. of $\phi(x)$

Now,



To be consistent:

$e^{iq\theta} = 1$

i.e. $\theta = \frac{2\pi k}{q}$

$U(1)^{(1)} \to \mathbb{Z}_q^{(1)}$

Wilson lines corresponding to probes with charges $\notin q\mathbb{Z}$ cannot end on $\phi(x)$ and have in fact a non trivial transformation under $\mathbb{Z}_q^{(1)}$

- In Maxwell theory there is no charged field, then WL for all probes have non-trivial $U(1)^{(1)}$ transformation.

- However, in YM there are ADJOINT FIELDS, i.e. the gluons gauge bosons.

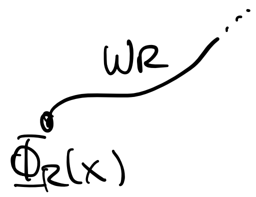
Probes in the adjoint rep produces WL that can end on the location of an adjoint field; then one can unlink the Σ_2 from the line and the corresponding WL must have zero charge

Only weights μ that are not in $\Lambda_{root}(G)$ give WL transforming non-trivially $\rightsquigarrow \mathbb{Z}_N^{(1)}$-sym.

Matter

Let us add matter in rep. R of G.

Now W_R can end on locations of matter fields



In general $g \in Z(G)^{(1)}$ acts on W_R (as g acts on R) while there is no top. linking between charge & charged ops
$\Rightarrow$ no top. op related to this g.
The only top. op remaining are those associated with elements of the subgroup
$$\Gamma_R = \{ g \in Z(G) \mid \phi_R(g) = 1 \} \subset Z(G)$$

Let us see this from a different perspective.
A matter field Φ is given by a family of locally defined Φ_i on each U_i, valued in R and glued in $U_i \cap U_j$ as
$$\Phi_j = \rho_R(g_{ij}) \cdot \Phi_i$$

A 1-form sym. transf. $g_{ij} \to g_{ij} t_{ij}$ affects Φ by modifying the gluing conditions unless $t_{ij} \in \Gamma_R$.