1) Consider the example of Lifetime utility maximization.
Solve it assuming $U(C) = \ln C, r = 1, \delta = 0, K(0) = 1$ and $T = 1$.

$$max \int_0^1 \ln C(t)\ dt$$
$$subject\ to$$
$$K' = K(t) - C(t)$$
$$K(0) = 1 \quad K(1) \geq 0$$

The Hamiltonian function is

$$H = \ln C(t) + \lambda(K(t) - C(t))$$

(i) $\frac{dH}{dC} = \frac{1}{C(t)} - \lambda = 0$

(ii) $K' = K(t) - C(t)$

(iii) $\lambda' = -\lambda$

(iv) $\lambda(1) \geq 0, K(1) \geq 0\ ,\ (K(1))\lambda(1) = 0$

(Truncated vertical line)

Solving (iii) we get $\lambda(t) = \lambda_0 e^{-t}$ where λ_0 is an arbitrary constant
(note that (iii) is an homogeneous differential equation with constant term and constant coefficient)
Replacing in (i) $\frac{1}{C(t)} = \lambda_0 e^{-t}$ → $C(t) = {\lambda_0}^{-1} e^t$
Then $C(t)$ is increasing in t if $\lambda_0 > 0$
Replacing in (ii) we get:

$$K' - K(t) = -{\lambda_0}^{-1} e^t$$

(note the general form of a first order linear differential equation is $\frac{dy}{dt} + u(t)y = w(t)$)
The general solution of this differential equation is given by

$$K(t) = e^{-\int u(t)dt}(A + \int w(t) e^{\int u(t)dt} dt)$$

where $(t) = -1$, $w(t) = -{\lambda_0}^{-1} e^t$ and $\int u(t)dt = -\int 1dt = -t - k$ where k is an arbitrary constant. Replacing into the expression above we have:

$$K = e^{t+k}(A - {\lambda_0}^{-1} \int e^{-k} dt)$$

$$K(t) = e^{t+k}(A - {\lambda_0}^{-1} e^{-k}(t + h)) \quad \text{where } h \text{ is an arbitrary constant}$$

$$K(t) = (Ae^{t+k} - {\lambda_0}^{-1} e^t (t + h))$$

Note that:

1) Given that $C(t) = {\lambda_0}^{-1} e^t$, λ_0 must be different from 0 otherwise $C(t)$ is not defined
2) given that $\lambda(t) = \lambda_0 e^{-t}$ for any $\lambda_0 \neq 0$ we have that for $t = 1$ → $\lambda(1) = \lambda_0 e^{-1} \neq 0$

Then the terminal condition has to be $K(1) = 0$
Using the initial and the terminal conditions:

$$K(0) = \left(Ae^k - {\lambda_0}^{-1} h\right) = 1$$

$$K(1) = e\left(Ae^{k} - \lambda_0{}^{-1}(1+h)\right) = 0$$

From the second equation we have $Ae^{k} = \lambda_0{}^{-1}(1+h)$ and replacing in the first one:

$\lambda_0{}^{-1}\big((1+h) - h\big) = 1$ → $\lambda_0 = 1$

Replacing $\lambda_0 = 1$in $K(t)$ we have:

$$K(t) = \left(Ae^{t+k} - e^{t}(t+h)\right) = e^{t}\big(Ae^{k} - t - h\big) = e^{t}(B - t)$$

where B is an arbitrary constant. Using Initial and terminal conditions we find that $B = 1$, then:

$$K(t) = e^{t}(1 - t)$$

Replacing $\lambda_0 = 1$in $C(t)$ the optimal consumption path:

$$C(t) = e^{t}$$

Replacing $\lambda_0 = 1$in $\lambda(t)$ we get the evolution of the shadow prices

$$\lambda(t) = e^{-t}$$

2) Consider the problem above in discrete time.
   a. Solve it with $T = 4$. (the function to maximize is $\sum_{t=0}^{4} U(C_t)$)

The problem is:

$$\max_{C_t} \sum_{t=0}^{4} \ln C_t$$

$$s.t.\ C_t = 2K_t - K_{t+1}$$
$$K_0 = 1$$

Note that:

i) $\delta = 0$ in continuous time is equivalent to a discount factor equal 1 in discrete time.

ii) the constraint $K' = K(t) - C(t)$ in discrete time becomes
$$K_{t+1} - K_t = K_t - C_t$$

iii) in the solution $K_5 = 0$ (no residual capital is left because utility function satisfies no satiation)

Replacing the constraint into the objective function:

$$\max_{K_t} \sum_{t=0}^{4} \ln(2K_t - K_{t+1})$$
$$0 \leq K_{t+1} \leq 2K_t$$
$$K_0 = 1$$

The first order condition is

$$-\frac{1}{2\,K_{t-1} - K_t} + \frac{2}{2\,K_t - K_{t+1}} = 0$$
$$-\frac{1}{C_{t-1}} + \frac{2}{C_t} = 0$$

that gives

$C_t = 2C_{t-1}$

We can explicit each C_t as

$C_t = 2^t C_0$.

$$x_0 = 1 = \sum_{t=0}^{4} \frac{c_t}{R^{t+1}} = \sum_{t=0}^{4} \frac{2^t C_0}{2^{t+1}} = \sum_{t=0}^{4} \frac{C_0}{2}$$
$$C_0 = \frac{2}{5}$$
$$C_t = \frac{2^{t+1}}{5}$$

b. Solve it with $T = \infty$.

The reduced problem now is

$$\max_{K_t} \sum_{t=0}^{\infty} \ln(2K_t - K_{t+1})$$
$$0 \le K_{t+1} \le 2K_t$$
$$K_0 = 1$$

The first order condition is

$$-\frac{1}{2\,K_{t-1} - K_t} + \frac{2}{2\,K_t - K_{t+1}} = 0$$
$$-\frac{1}{C_{t-1}} + \frac{2}{C_t} = 0$$

that gives

$C_t = 2C_{t-1}$

We can explicit each C_t as

$C_t = 2^t C_0$.

$$x_0 = 1 = \sum_{t=0}^{\infty} \frac{c_t}{R^{t+1}} = \sum_{t=0}^{\infty} \frac{2^t C_0}{2^{t+1}} = \sum_{t=0}^{\infty} \frac{C_0}{2}$$

Cannot be satisfied for every positive value of C_0. Then, no solution.

3) Maximize $\int_0^T \ln(q)\, e^{-\delta t} dt$ subject to $s' = -q$ and $S(0) = s_0,\ S(T) \geq 0$

$$H = \ln(q)\, e^{-\delta t} + \lambda(-q)$$

(i) $\frac{dH}{dC} = \frac{1}{q} e^{-\delta t} - \lambda = 0$

(ii) $s' = -q$

(iii) $\lambda' = 0$

(iv) $\lambda(T) \geq 0, s(T) \geq 0\ ,\ (s(T))\lambda(T) = 0$

From (iii) $\lambda(t) = \lambda_0 \quad \forall t$ (where λ_0 is constant)

From (i) we have:

$\frac{1}{q} e^{-\delta t} - \lambda_0 = 0 \rightarrow q = \frac{1}{\lambda_0} e^{-\delta t}$

Replacing q in (ii) we get:

$$s' = -\frac{1}{\lambda_0} e^{-\delta t}$$

$$s = -\int \frac{1}{\lambda_0} e^{-\delta t}\, dt = \frac{1}{\delta\lambda_0} e^{-\delta t} + h$$

Note λ_0 has to be different from zero otherwise q and s are not defined.

Therefore condition (iv) has to be $s(T) = 0$

$$h = -\frac{1}{\delta\lambda_0} e^{-\delta T} \qquad [1]$$

Using initial condition we have:

$$s_0 = \frac{1}{\delta\lambda_0} + h$$

Replacing h using [1] and solving by λ_0 we get:

$$s_0 = \frac{1}{\delta\lambda_0}(1 - e^{-\delta T})$$

$$\lambda_0 = \frac{1}{\delta s_0}(1 - e^{-\delta T})$$

Replacing λ_0 in [1] we get

$$h = -\frac{1}{\delta\lambda_0} e^{-\delta T} = \frac{e^{-\delta T}}{(1 - e^{-\delta T})} s_0$$

Replacing in the expression fot s:

$$s = \frac{1}{\delta\lambda_0} e^{-\delta t} + h = \frac{e^{-\delta t}}{1 - e^{-\delta T}} s_0 + \frac{e^{-\delta T}}{1 - e^{-\delta T}} s_0 = \frac{e^{-\delta t} + e^{-\delta T}}{1 - e^{-\delta T}} s_0$$

Section 20.2

Ex. 1

$$H = 6y + \lambda(y + u)$$

Conditions are:

1. $\frac{dH}{du} = \lambda$
2. $y' = y + u$
3. $\lambda' = -6 - \lambda$
4. $\lambda(8) = 0$
5. $y(0) = 10$

The general solution of 3 is

$$\lambda(t) = Ae^{-t} - 6$$

We can use the trasversality condition to find the definite solution

$$\lambda(8) = Ae^{-8} - 6 = 0$$

$$A = 6e^{8}$$

The definite solution is

$$\lambda(t) = 6e^{8-t} - 6$$

Then $> 0 \; \forall \; 0 \leq t \leq 8$.

By condition 1 we can say that the Hamiltonian is maximized for $u = 2 \; \forall \; 0 \leq t \leq 8$

Replacing in 2 we have

$$y' = y + 2$$

The general solution of is

$$y(t) = Ae^{t} - 2$$

Using the initial condition $y(0) = 10$ we find the definite solution:

$$y(0) = Ae^{0} - 2 = 10$$

$$A = 12$$

That is:

$$y(t) = 12e^{t} - 2$$