4 Dicembre

$L[y] = y'' + by' + cy = 0 \qquad y = e^{rx}$

$L[e^{rx}] = p(r)\, e^{rx} = 0 \qquad p(r) = r^2 + br + c$

$L[y] = \left(\underbrace{\left(\frac{d^2}{dx^2}\right) + b\left(\frac{d}{dx}\right) + c}_{p\left(\frac{d}{dx}\right)}\right) y \qquad \frac{d}{dx} \rightsquigarrow r$

$= \left(\left(\frac{d}{dx}\right)^2 + b\left(\frac{d}{dx}\right)^1 + c\left(\frac{d}{dx}\right)^0\right) y$

$p(r) = r^2 + br + c$

$p(r) = r^2 + br + c = 0$

$r_\pm = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$

se $b^2 - 4c \neq 0 \quad r_+ \neq r_-$ e restano

definite due funzioni $e^{r_+ x}$ e $e^{r_- x}$

$y_h = A e^{r_+ x} + B e^{r_- x} \qquad A, B \in \mathbb{C}$

Soluzione generale dell'equazione omogenea

In generale se $z = x + iy \in \mathbb{C}$ è ben definita e^z. Valgono

1) $e^{z_1 + z_2} = e^{z_1} e^{z_2}$

2) $(e^{z_1})^{z_2} = e^{z_1 z_2}$

In particolare

$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$

$e^{iy} = \cos(y) + i \sin(y)$

$e^{-iy} = \cos(-y) + i \sin(-y) = \cos(y) - i \sin y$

$\begin{cases} e^{iy} = \cos(y) + i \sin(y) \\ e^{-iy} = \cos(y) - i \sin(y) \end{cases}$

Sommandole

$2\cos y = e^{iy} + e^{-iy} \iff \cos y = \frac{e^{iy} + e^{-iy}}{2}$

Sottraendo la seconda alla prima

$2i \sin(y) = e^{iy} - e^{-iy} \iff \sin y = \frac{e^{iy} - e^{-iy}}{2i}$

Formule di Euler.

$$\begin{cases} e^{iy} = \cos(y) + i\sin(y) \\ e^{-iy} = \cos(y) - i\sin(y) \end{cases}$$

Sommandole

$$2\cos y = e^{iy} + e^{-iy} \Longleftrightarrow \cos y = \frac{e^{iy}+e^{-iy}}{2}$$

sottraendo la seconda alla prima

$$2i\sin(y) = e^{iy} - e^{-iy} \Longleftrightarrow \sin y = \frac{e^{iy}-e^{-iy}}{2i}$$

Formule di Euler.

$$\sqrt{-3} = i\sqrt{3}$$

$$L[y] = y'' - y' + y = 0$$

$$r^2 - r + 1 = 0 \qquad r_\pm = \frac{1}{2} \pm \frac{\sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y_h = A e^{r_+ x} + B e^{r_- x} =$$

$$= A e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + B e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}$$

$$= A e^{\frac{1}{2}x} e^{i\frac{\sqrt{3}}{2}x} + B e^{\frac{1}{2}x} e^{-i\frac{\sqrt{3}}{2}x}$$

$$= e^{\frac{x}{2}}\left(A e^{i\frac{\sqrt{3}}{2}x} + B e^{-i\frac{\sqrt{3}}{2}x}\right)$$

$$\sin y = \frac{e^{iy}-e^{-iy}}{2i} \qquad \cos y = \frac{e^{iy}+e^{-iy}}{2} \qquad y = \frac{\sqrt{3}}{2}x$$

$$\frac{1}{2} e^{i\frac{\sqrt{3}}{2}x} + \frac{1}{2} e^{-i\frac{\sqrt{3}}{2}x} = \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\frac{1}{2i} e^{i\frac{\sqrt{3}}{2}x} - \frac{1}{2i} e^{-i\frac{\sqrt{3}}{2}x} = \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Anche $e^{\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x$, $e^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$ sono soluzioni

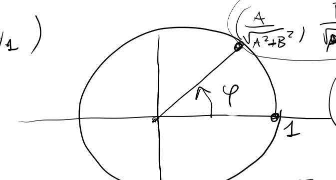
$$y_h = A e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) + B e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right) = \qquad (A,B) \neq (0,0)$$

$$= e^{\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$= a e^{\frac{x}{2}}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{B}{\sqrt{A^2+B^2}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \qquad a = \sqrt{A^2+B^2}$$

(x_1, y_1)

$\sqrt{x_1^2 + y_1^2}$



$$\left(\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right) = (\cos\varphi, \sin\varphi)$$

$$\left(\frac{A}{\sqrt{A^2+B^2}}\right)^2 + \left(\frac{B}{\sqrt{A^2+B^2}}\right)^2 = \frac{A^2}{A^2+B^2} + \frac{B^2}{A^2+B^2} = 1$$

$$= a e^{\frac{x}{2}}\left(\cos\varphi\cos\left(\frac{\sqrt{3}}{2}x\right) + \sin\varphi\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$= \boxed{a e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x + \varphi\right)} \qquad \vartheta = -\frac{\sqrt{3}}{2}\varphi$$

$$= a e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}(x - \vartheta)\right)$$

$$L[y] = y'' + by' + cy = 0$$

$$P(r) = 0$$

$$r^2 + br + c = 0 \implies r_+ = r_- = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$$

$\sqrt{b^2-4c} = 0$

$b^2 = 4c$

$y = e^{-\frac{b}{2}x}$ resta una

soluzione $\quad y = u\, e^{-\frac{b}{2}x} = u\, e^{r_+ x}$ con u

non costante

$$L[u e^{r_+x}] = (u e^{r_+x})'' + b(u e^{r_+x})' + c\, u\, e^{r_+x}$$

$$(u e^{r_+x})' = u' e^{r_+x} + u (e^{r_+x})'$$

$$(u e^{r_+x})'' = ((u e^{r_+x})')' = (u' e^{r_+x} + u (e^{r_+x})')'$$

$$= u'' e^{r_+x} + u'(e^{r_+x})' + u'(e^{r_+x})' + u(e^{r_+x})''$$

$$= u'' e^{r_+x} + 2u'(e^{r_+x})' + u(e^{r_+x})''$$

$$L[u e^{r_+ x}] = (u e^{r_+ x})'' + b\,(u e^{r_+ x})' + c\, u\, e^{r_+ x}$$

$$(u e^{r_+ x})' = u' e^{r_+ x} + u\,(e^{r_+ x})'$$

$$(u e^{r_+ x})'' = ((u e^{r_+ x})')' = \left(u' e^{r_+ x} + u\,(e^{r_+ x})'\right)'$$

$$= u'' e^{r_+ x} + u'(e^{r_+ x})' + u'(e^{r_+ x})' + u\,(e^{r_+ x})''$$

$$= u'' e^{r_+ x} + 2u'(e^{r_+ x})' + u\,(e^{r_+ x})''$$

$$= u'' e^{r_+ x} + u'\left(2(e^{r_+ x})' + b\, e^{r_+ x}\right) + u\, \underline{L[e^{r_+ x}]}$$

$$= u'' e^{r_+ x} + u' \underbrace{(2 r_+ + b)}_{p'(r_+)} e^{r_+ x} + u \underbrace{p(r_+)}_{0} e^{r_+ x}$$

$p(r) = r^2 + br + c$ $\qquad$ $p(r_+) = 0$

$p'(r) = 2r + b$ $\qquad$ $p'(r_+) = 0$

Siccome $\Delta = b^2 - 4c = 0$

$$p(r) = (r - r_+)^2$$

$$p'(r) = 2(r - r_+)\Big|_{r=r_+} = 0$$

Se scelgo una funzione u qualsiasi $e^{r_+ x}$

allora

$$L[u e^{r_+ x}] = u'' e^{r_+ x} = 0$$

$u'' = 0$ $\qquad$ $u = x$

$x e^{r_+ x}$

$$y_h = A e^{r_+ x} + B x e^{r_+ x}$$

$$y'' - 3y' + 2y = 0 \qquad p(r) = r^2 - 3r + 2 = (r-1)(r-2), \quad r = 1, 2$$

$$y_h = A e^{x} + B e^{2x}$$

$$y'' - y' + 2y = 0 \qquad p(r) = r^2 - r + 2 = 0$$

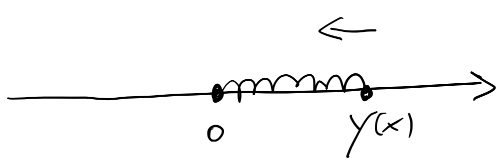
$$r_\pm = \frac{1}{2} \pm \frac{\sqrt{1-8}}{2} = \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y_h = A e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right) + B e^{\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right)$$

$$y'' + 2y = 0 \qquad p(r) = r^2 + 2 = 0, \quad r = \pm i\sqrt{2}$$

$$y_h = A\cos(\sqrt{2}\,x) + B\sin(\sqrt{2}\,x)$$

$$y'' = -\frac{1}{100}y' - y$$

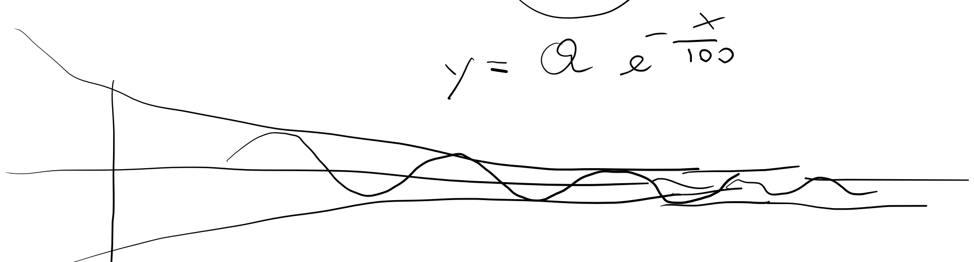


$$y'' + \frac{1}{10^2}y' + y = 0$$

$$p(r) = r^2 + \frac{r}{10^2} + 1 = 0 \qquad r_\pm = -\frac{1}{200} \pm \frac{\sqrt{10^{-4} - 4}}{2} = -\frac{1}{200} \pm \frac{i\sqrt{4 - 10^{-4}}}{2}$$

$$y = a\, e^{-\frac{x}{100}} \cos\left(\frac{\sqrt{4-10^{-4}}}{2}x + \varphi_0\right)$$

$$y = a\, e^{-\frac{x}{100}}$$



$$y'' - 2y' + y = 0$$

$$p(r) = r^2 - 2r + 1 = (r-1)^2 \qquad r = 1$$

$$y = (A + Bx)e^x$$

$$y'' + by' + cy = f(x)$$

$$L[y] = f(x) \qquad \ast$$

Se y_1 e y_2 sono soluzioni di $\ast$ allora

$$L[y_1 - y_2] = L[y_1] - L[y_2] = f - f = 0$$

Per trovare la soluzione generale di $\ast$ basta scriverla come $y_g = y_h + y_p$ dove

y_g è la soluzione generale della $\ast$ (non omogenea)

y_h " " " " " omogenea associata

y_p è una qualsiasi soluzione della $\ast$ (non omogenea)

Applicheremo il metodo della somiglianza per trovare y_p

$$L[y] = f$$

$$y'' + by' + cy = f(x) \qquad b, c \in \mathbb{R}$$

$$f(x) = e^{\alpha x} \qquad \alpha \in \mathbb{C}$$

$$= e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x)$$

$$= p(x)\, e^{\alpha x}\cos(\beta x), \quad p(x)\, e^{\alpha x}\sin(\beta x)$$

$$y'' + by' + cy = 2e^{\alpha x}$$

$$y_p = C e^{\alpha x}.$$

$$L[y_p] = 2e^{\alpha x}$$

$$L[Ce^{\alpha x}] = C\, L[e^{\alpha x}] =$$

$$= C\, p(\alpha)\, e^{\alpha x} = 2e^{\alpha x}$$

$$C\,p(\alpha) = 2$$

$$C = \frac{2}{p(\alpha)} \qquad \text{se } p(\alpha) \neq 0$$

Se $p(\alpha) = 0$ cerchiamo una soluzione della forma $y_p = C\, x\, e^{\alpha x}$

$$L[y_p] = 2e^{\alpha x}$$

$$C\, L[xe^{\alpha x}] = 2e^{\alpha x}$$

$$= C\,(p'(\alpha)\, e^{\alpha x} + x \underbrace{p(\alpha)}_{0} e^{\alpha x}) = 2e^{\alpha x}$$

$$= C\, p'(\alpha)\, e^{\alpha x} = 2e^{\alpha x} \qquad C = \frac{2}{p'(\alpha)}$$

se $p'(\alpha) \neq 0$