28th of November

Theorem (convolution) Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ $\quad p,q \in [1,+\infty]$

set

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

Then (Young's convolution inequality)

$$|f*g|_{L^r(\mathbb{R}^d)} \le |f|_{L^p(\mathbb{R}^d)}\, |g|_{L^q(\mathbb{R}^d)}$$

for $\quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

Proof

$$L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$$

$T : X \to Y$

$$\sup \left\{ \|Tx\|_Y : \|x\|_X \le 1,\ x \in D \text{ where } D \text{ is dense in } X \right\}$$

We will assume that $C_c^0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$

$\forall \quad 1 \le p < +\infty$.

$$\int \underbrace{\left(\int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right)}_{L^r(\mathbb{R}^d)} h(x)\, dx \qquad h \in L^{r'}(\mathbb{R}^d)$$

$$I(f,g,h) = \int f(x-y)\, g(y)\, h(x)\, dx\, dy$$

It is enough to prove $\quad I(f,g,h) \le 1 \quad$ if

$$|f|_{L^p(\mathbb{R}^d)} = |g|_{L^q(\mathbb{R}^d)} = |h|_{L^{r'}(\mathbb{R}^d)} = 1$$

$f \ge 0, \quad g \ge 0, \quad h \ge 0 \qquad \frac{1}{r'} = 1 - \frac{1}{r} \qquad \frac{1}{r} = 1 - \frac{1}{r'}$

if $p < +\infty \quad f \in C_c^0(\mathbb{R}^d) \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$q < +\infty \quad g \in C_c^0(\mathbb{R}^d) \qquad 2 = \frac{1}{r'} + \frac{1}{p} + \frac{1}{q}$

$r' < +\infty \quad h \in C_c^0(\mathbb{R}^d)$

$$\left(2 - \frac{1}{p} - \frac{1}{q}\right) r' = 1$$

$$\left(2 - \frac{1}{p} - \frac{1}{r'}\right) q = 1 \qquad \left(2 - \frac{1}{r'} - \frac{1}{q}\right) p = 1$$

$$\begin{cases} \left(1 - \frac{1}{p}\right) r' + \left(1 - \frac{1}{q}\right) r' = 1 \\ \left(1 - \frac{1}{p}\right) q + \left(1 - \frac{1}{r'}\right) q = 1 \\ \left(1 - \frac{1}{r'}\right) p + \left(1 - \frac{1}{q}\right) p = 1 \end{cases} \qquad \frac{1}{r} p + \frac{1}{q'} p = 1$$

$$I(f,g,h) = \int f(y)\, g(x-y)\, h(x)\, dx\, dy =$$

$$= \int \left(f^p(y)\, g^q(x-y)\right)^{1-\frac{1}{r'}} \left(f^p(y)\, h^{r'}(x)\right)^{1-\frac{1}{q}} \left(g^q(x-y)\, h^{r'}(x)\right)^{1-\frac{1}{p}} dx\, dy$$

$$\frac{1}{r} + \frac{1}{p'} + \frac{1}{q'} = 1$$

$$\le \left| \left(f^p(y)\, g^q(x-y)\right)^{\frac{1}{r}} \right|_{L^r(\mathbb{R}^d \times \mathbb{R}^d)} \left| \left(f^p(y)\, h^{r'}(x)\right)^{\frac{1}{q'}} \right|_{L^{q'}(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$\left| \left(g^q(x-y)\, h^{r'}(x)\right)^{\frac{1}{p'}} \right|_{L^{p'}(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$= \left| f^p(y)\, g^q(x-y) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{r}} \left| f^p(y)\, h^{r'}(x) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{q'}}$$

$$= \left| g^q(x-y)\, h^{r'}(x) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbb{R}^d} dy\, \overbrace{f^p(y)} \int_{\mathbb{R}^d} \overbrace{g^q(x-y)\, dx} \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^d} f^p(y)\, dy \int_{\mathbb{R}^d} h^{r'}(x)\, dx \right)^{\frac{1}{q'}}$$

$$\cdot \left(\int_{\mathbb{R}^d} dx\, h^{r'}(x) \int_{\mathbb{R}^d} g^q(x-y)\, dy \right)^{\frac{1}{p'}}$$

$$= |f|_{L^p}^{\frac{p}{r}}\, |g|_{L^q}^{\frac{q}{r}}\, |f|_{L^q}^{\frac{p}{q'}}\, |h|_{L^{r'}}^{\frac{r'}{q'}}$$

$$|h|_{L^{r'}}^{\frac{r'}{p'}}\, |g|_{L^q}^{\frac{q}{p'}} \qquad p\left(\frac{1}{r} + \frac{1}{q'}\right) = 1$$

$$= 1$$

$p < +\infty, \quad q < +\infty \quad r' < +\infty$

$$I : \overbrace{C_c^0(\mathbb{R}^d)}^{L^p} \times \overbrace{C_c^0(\mathbb{R}^d)}^{L^q} \times \overbrace{C_c^0(\mathbb{R}^d)}^{L^{r'}} \to \mathbb{R}$$

$$|I(f,g,h)| \le |f|_{L^p}\, |g|_{L^q}\, |h|_{L^{r'}}$$

Proof $f \in C_c^k(\mathbb{R}^d)$ $g \in L^1_{loc}(\mathbb{R}^d)$

Then $f * g \in C^k(\mathbb{R}^d)$

$$\partial_x^\alpha (f * g) = (\partial_x^\alpha f) * g$$

$\forall$ $|\alpha| \leq k$

$$f * g = g * f$$

Theorem $\quad \rho \in L^1(\mathbb{R}^d) \qquad \int_{\mathbb{R}^d} \rho(x)\,dx = 1$

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right) \quad \forall \varepsilon > 0$$

For any $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < +\infty$

we have $\displaystyle\lim_{\varepsilon \to 0^+} \rho_\varepsilon * f = f$ in $L^p(\mathbb{R}^d)$

$\Downarrow$

In particular $\displaystyle\lim_{t \to 0^+} e^{t\Delta} f = f$ in $L^p(\mathbb{R}^d)$

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

$\rho_{\sqrt{t}} \qquad \rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$

$$\begin{cases} (\partial_t - \Delta) u = 0 & t > 0 \\ u(0) = f \in L^\infty(\mathbb{R}) \end{cases}$$

$p < +\infty \qquad f \in C^0_c(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$

$$\rho_\varepsilon * f(x) - f(x) =$$

$$= \int_{\mathbb{R}^d} f(x-y)\, \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) dy - \int_{\mathbb{R}} f(x)\, \rho(z)\, dz$$

$$z = \frac{y}{\varepsilon}$$

$$= \int_{\mathbb{R}^d} \Big(f(x-\varepsilon z) - f(x) \Big) \rho(z)\, dz$$

$$\| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} \left(f(\cdot - \varepsilon z) - f \right) \rho(z)\, dz \right\|_{L^p(\mathbb{R}^d)}$$

$$\leq \int_{\mathbb{R}^d} \underbrace{\| f(\cdot - \varepsilon z) - f \|_{L^p(\mathbb{R}^d)}}_{\Delta(\varepsilon z)} |\rho(z)|\, dz$$

$$= \int_{\mathbb{R}^d} \Delta(\varepsilon z) |\rho(z)|\, dz$$

$$\lim_{y \to 0} \Delta(y) = 0 \qquad \Delta(y) = \| f(\cdot - y) - f \|_{L^p(\mathbb{R}^d)}$$

$$\left| \left(f(x-y) - f(x) \right) \right|^p =$$

$$= \left| 1_K(x) \left(f(x-y) - f(x) \right) \right|^p \leq \underline{C\, 1_K(x)}$$

$$\lim_{y \to 0} \| f(\cdot - y) - f \|^p_{L^p(\mathbb{R}^d)} = \lim_{y \to 0} \int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx$$

$$= \int_{\mathbb{R}} \underbrace{\lim_{y \to 0} |f(x-y) - f(x)|^p}_{0} dx \qquad |\Delta(y)| \leq 2 \|f\|_{L^p(\mathbb{R}^d)}$$

$$|\Delta(\varepsilon z) \rho(z)| \leq 2 \|f\|_{L^p}\, \rho(z)$$

$$\| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} \left(f(\cdot - \varepsilon z) - f \right) \rho(z)\, dz \right\|_{L^p(\mathbb{R}^d)}$$

$$\leq \int_{\mathbb{R}^d} \underbrace{\| f(\cdot - \varepsilon z) - f \|_{L^p(\mathbb{R}^d)}}_{\Delta(\varepsilon z)} |\rho(z)|\, dz$$

$$= \int_{\mathbb{R}^d} \Delta(\varepsilon z) |\rho(z)|\, dz$$

$$0 \leq \underbrace{\lim_{\varepsilon \to 0} \| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)}}_{= 0} \leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \Delta(\varepsilon z) |\rho(z)|\, dz$$

$$\int_{\mathbb{R}^d} \underbrace{\lim_{\varepsilon \to 0} \Delta(\varepsilon z)}_{0} |\rho(z)|\, dz = 0$$

If $f \in C_c^0(\mathbb{R}^d)$. If $f \in L^p(\mathbb{R}^d) \setminus C_c^0(\mathbb{R}^d)$

we pick $\tilde{f} \in C_c^0(\mathbb{R}^d)$ s.t. $\| f - \tilde{f} \|_{L^p(\mathbb{R}^d)} \leq \delta$ with $\delta > 0$.

$$\| \rho_\varepsilon * f - f \|_{L^p} = \| \rho_\varepsilon * (f - \tilde{f}) + \rho_\varepsilon * \tilde{f} - \tilde{f} + \tilde{f} - f \|_{L^p}$$

$$\leq \| \rho_\varepsilon * (f - \tilde{f}) \|_{L^p} + \underbrace{\| \rho_\varepsilon * \tilde{f} - \tilde{f} \|_{L^p}}_{\xrightarrow{\varepsilon \to 0} 0} + \underbrace{\| \tilde{f} - f \|_{L^p}}_{< \delta}$$

$$\leq \underbrace{\| \rho_\varepsilon \|_{L^1}}_{\| \rho \|_{L^1}} \| f - \tilde{f} \|_{L^p}$$

$$\frac{1}{p} + 1 = 1 + \frac{1}{p} \qquad r = p \qquad q = 1$$

$$\| f * g \|_{L^r} \leq \| f \|_{L^p} \| g \|_{L^q} \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

$$\limsup_{\varepsilon \to 0} \| \rho_\varepsilon * f - f \| \leq 2\delta \qquad \forall \delta > 0$$

$k \in L^q(\mathbb{R}^d)$

$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$f \to Tf = k * f$

$L^p(\mathbb{R}^d) \longrightarrow L^r(\mathbb{R}^d)$

Hörmander

$r \geq p$

$\forall y \in \mathbb{R}^d$

$\tau_y f(x) = f(x-y)$

$\tau_y T = T \tau_y$

Theorem (K R F)

Let $\mathcal{F} \subseteq L^p(\mathbb{R}^d)$ $p<+\infty$

be a bounded subset of $L^p(\mathbb{R}^d)$ st. $\tau_h f = f(\cdot - h)$

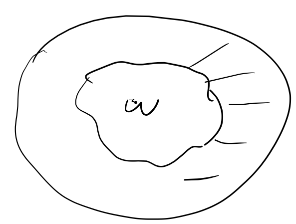
(a) $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0$ st $|h| < \delta(\varepsilon) \Rightarrow |\tau_h f - f|_{L^p(\mathbb{R})} < \varepsilon \quad \forall f \in \mathcal{F}$.

Then $\forall\ \Omega \subset \mathbb{R}^d$ open bounded $\mathcal{F}|_{\Omega}$ is relatively compact in $L^p(\Omega)$.

Pf We need ~~need~~ to prove that $\forall \varepsilon > 0$ $\mathcal{F}|_{\Omega}$ is in a finite union of balls of radius ε in $L^p(\Omega)$.

(1) $\exists\ \omega \subset\subset \Omega$ st

$|f|_{L^p(\Omega \setminus \omega)} \leq \frac{\varepsilon}{3} \quad \forall f \in \mathcal{F}$



Ω

$T(a,b) = \{ f \in C^1(\mathbb{R}^d) : |f|_{L^\infty(\mathbb{R}^d)} \leq a,\ |\nabla f|_{L^\infty(\mathbb{R}^d)} \leq b \}$

$T(a,b)|_{\omega}$ is relatively compact $C^0(\omega)$

$\rho \in C_c^\infty(D_{\mathbb{R}^d}(0,1), [0,+\infty))$ $\rho_n(x) = \underline{n^d \rho(nx)}$

$D_{\mathbb{R}^d}(0, \frac{1}{n})$

$n > \frac{1}{\delta(\frac{\varepsilon}{4})}$ $\frac{1}{n} < \delta(\frac{\varepsilon}{4})$

$$|\rho_n * f - f|_{L^p(\mathbb{R}^d)} = \Big| \int_{\mathbb{R}^d} \rho_n(y) \big(\overbrace{f(x-y)}^{\tau_y f(x)} - f(x) \big) dy \Big|_{L^p(\mathbb{R}^d_x)} \leq$$

$$\leq \int_{|y|<\delta(\frac{\varepsilon}{4})} \rho_n(y) \underbrace{|\tau_y f - f|_{L^p(\mathbb{R}^d)}}_{\frac{\varepsilon}{4}} dy \leq \int_{\mathbb{R}^d} \rho_n(y)\, dy\ \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$$

$\forall f \in \mathcal{F}$

$|\rho_n * f(x)| \le a_n \quad \forall f \in \mathcal{F}$

$|\rho_n * f(x)| = \left|\int_{\mathbb{R}^d} \rho_n(x-y) f(y)\, dy\right| \le$

$\le \int_{\mathbb{R}^d} \rho_n(x-y)\, |f(y)|\, dy \le |\rho_n|_{L^{p'}} |f|_{L^p} \quad \forall f \in \mathcal{F}$

$\le C\, |\rho_n|_{L^{p'}}$

$|\nabla(\rho_n * f)(x)| \le b_n \quad \forall f \in \mathcal{F}$

$\{\rho_n * f\}_{f \in \mathcal{F}} \subseteq T(a_n, b_n)$

There exists a finite family of

$\forall \varepsilon > 0$, $C_1 > 0$, $\mathbb{R}^d$, $\exists N$ and $u_1, \dots, u_N \in C^0(\omega)$ s.t.

$T(a_n, b_n)|_\omega \subseteq \bigcup_{j=1}^{N} D_{C^0(\omega)}\left(u_j, \frac{\varepsilon}{C_1}\right)$

$\cap$ $C^0(\omega)$

$C^0(\omega) \subseteq L^p(\omega)$

$T(a_n, b_n)|_\omega \subseteq \bigcup_{j=1}^{N} D_{L^p(\omega)}\left(u_j, \frac{\varepsilon}{3}\right)$

$\mathcal{F}|_\Omega \subseteq \bigcup_{j=1}^{N} D_{L^p(\Omega)}(u_j, \varepsilon)$

$C^0(\omega) \xhookrightarrow{j} L^p(\omega) \qquad |g|_{L^p} \le |g|_{L^\infty} |\omega|^{\frac{1}{p}}$

$g \longrightarrow g \qquad \left(\int_\omega |g|^p\, dx\right)^{\frac{1}{p}} \le |g|_{L^\infty} \left(\int_\omega 1\, dx\right)^{\frac{1}{p}}$

$\|j\|_{C^0(\omega) \to L^p(\omega)} \le |\omega|^{\frac{1}{p}} \le |\Omega|^{\frac{1}{p}}$

$D_{C^0(\omega)}\left(u_j, \frac{\varepsilon}{C_1}\right) \subseteq D_{L^p(\omega)}\left(u_j, \frac{\varepsilon |\Omega|^{\frac{1}{p}}}{C_1}\right)$

$C_1 = 3|\Omega|^{\frac{1}{p}}$

$= D_{L^p(\omega)}\left(u_j, \frac{\varepsilon}{3}\right)$

We found $\forall \varepsilon > 0$, an ~~number~~ $\omega \subset\subset \Omega$

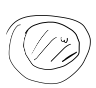
$\exists N$ and $u_1, \dots, u_N \in C^0(\omega)$

s.t. $\{\rho_n * f : f \in \mathcal{F}\} \subseteq \bigcup_{j=1}^{N} D_{L^p(\omega)}\left(u_j, \frac{\varepsilon}{3}\right)$

for $n >> 1$ and ω appropriate

$\Longrightarrow \{f|_\Omega : f \in \mathcal{F}\} \subseteq \bigcup_{j=1}^{N} D_{L^p(\Omega)}(u_j, \varepsilon)$

$f \in \mathcal{F}, \quad \rho_n * f|_\omega$



Ω

$|\rho_n * f - u_j|_{L^p(\omega)} < \frac{\varepsilon}{3}$

$|f - u_j|_{L^p(\Omega)} = |(f - \rho_n * f) + (\rho_n * f - u_j)|_{L^p(\Omega)}$

$\le |f - \rho_n * f|_{L^p(\Omega)} + |\rho_n * f - u_j|_{L^p(\Omega)}$

$\underbrace{|f - \rho_n * f|_{L^p(\mathbb{R}^d)}}_{< \frac{\varepsilon}{4}} \qquad \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

$|(\rho_n * f - u_j) 1_\omega + (\rho_n * f - u_j) 1_{\Omega \setminus \omega}|_{L^p(\Omega)}$

$\le \underbrace{|\rho_n * f - u_j|_{L^p(\omega)}}_{< \frac{\varepsilon}{3}} + \boxed{\underbrace{|\rho_n * f|_{L^p(\Omega \setminus \omega)}}_{< \frac{\varepsilon}{3}}}$

$\cancel{|\rho_n * f - f + f|_{L^p(\Omega \setminus \omega)}} \le$

$\le \cancel{\underbrace{|\rho_n * f - f|_{L^p(\mathbb{R}^d)}}_{< \frac{\varepsilon}{4}}} + \cancel{|f|_{L^p(\Omega \setminus \omega)}}$

$\forall n \; \exists \; \omega \subset\subset \Omega$ s.t. $|\rho_n * f|_{L^p(\Omega \setminus \omega)} < \frac{\varepsilon}{3}$

$|\rho_n * f|_{L^p(\Omega \setminus \omega)} \le \underbrace{|\rho_n * f|_{L^\infty(\mathbb{R}^d)}}_{\le a_n \; \forall f \in \mathcal{F}} |\Omega \setminus \omega|^{\frac{1}{p}} < \frac{\varepsilon}{3} \quad \forall \varepsilon > 0$

Ω

$|\Omega| = \sup_{\omega \subset\subset \Omega} |\omega| \Longrightarrow \forall \varepsilon > 0 \; \exists \; \omega \subset\subset \Omega$

s.t. $|\Omega \setminus \omega| = |\Omega| - |\omega| < \left(\frac{\varepsilon}{3 a_n}\right)^p$

Hilbert spaces $\qquad T \in \mathcal{L}(X)$

Def $\qquad e^{tT} \quad t \to +\infty$

H is a Pre-Hilbert space on $\mathbb{R}$.

Vector space on $\mathbb{R}$ and it has an inner product, that is a symmetric bilinear map

$$(u, v) \in H \times H \longrightarrow (u, v)_H \in \mathbb{R}$$

with $(u, u)_H > 0$ if $u \neq 0$.

$$\|u\|_H = \sqrt{(u, u)_H}$$

this defines a ~~notion~~ norm.

H is a Hilbert space if it is a B space for this norm.

$$L^2(X, d\mu) \qquad |f|_{L^2(X)} = \left(\int_X |f(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}}$$

$$(f, g)_{L^2(X)} = \int_X f(x) \, \overline{g(x)} \, d\mu(x)$$

$\forall\, a, b \in H$

$$\left\| \frac{a+b}{2} \right\|^2 + \left\| \frac{a-b}{2} \right\|^2 = \frac{\|a\|^2 + \|b\|^2}{2}$$

$$\| a+b \|^2 + \| a-b \|^2 = 2 \left(\|a\|^2 + \|b\|^2 \right)$$

$$|(a,b)_H| \leq \|a\| \, \|b\|$$

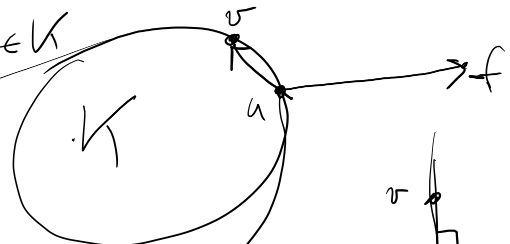
$$X \xrightarrow{T} Y$$

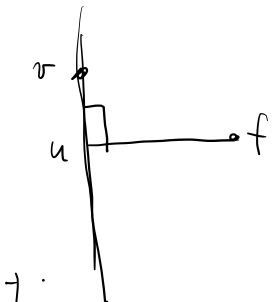
Teorem Let H be a Hilbert space and let $K \subseteq H$ be a closed nonempty convex set. Then $\forall\ f \in H\ \exists!\ u \in K$ s.t.

$$\|u-f\| \leq \|v-f\| \quad \forall v \in K$$

Furthermore u is characterized by

$$(f-u, v-u)_H \leq 0 \quad \forall\ v \in K$$





Pf $\phi(x) = \|x-f\|$ convex, strongly continuous

$$\lim_{x\to\infty} \phi(x) = +\infty \qquad \phi : K \to [0,+\infty)$$

$\exists\ \{x_n\}$ K s.t. $\lim_{n\to+\infty} \overbrace{\|x_n - f\|}^{d_n} = \inf\{\|x-f\| : x \in K\}$

$=: d \geq 0$

$n, m \qquad f - x_n \qquad f - x_m$

$$d^2 \leq \Big\|f - \underbrace{\frac{x_n+x_m}{2}}_{\in K}\Big\|^2 + \Big\|\frac{x_n-x_m}{2}\Big\|^2 =$$

$$= \frac{1}{2}\left(\|f-x_n\|^2 + \|f-x_m\|^2\right)$$

$$= \left(\Big\|\frac{f-x_n + f-x_m}{2}\Big\|^2 + \Big\|\frac{(f-x_n)-(f-x_m)}{2}\Big\|^2 \right)$$

$$\Big\|\frac{x_n - x_m}{2}\Big\|^2 \leq \frac{1}{2}\left(d_n^2 + d_m^2\right) - d^2 \xrightarrow[n,m\to\infty]{} 0$$

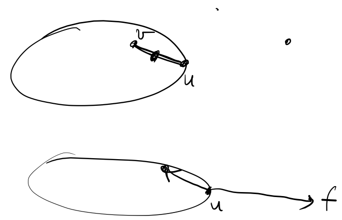
$\forall\ n, m >> 1$

$\Rightarrow \|x_n - x_m\|$ is as small as we want., so $\{x_n\}$ is Cauchy $\quad K$

$$\exists\ u = \lim_{n\to+\infty} x_n \text{ in } K.$$

$$d = \lim_{n\to+\infty} \|x_n - f\| = \|u-f\|$$

Let us prove that



$(f-u, v-u) \leq 0$ $\quad *$

$\forall v \in K$

$t \in [0,1]$

$h(t) = \| f - (\underbrace{(1-t)u + tv}) \|^2 = \| (f-u) - (v-u)t \|^2 =$

$= \| f-u \|^2 - 2t(f-u, v-u) + t^2 \| v-u \|^2$

$t=0$ is an abs minimum: $\Rightarrow h'(0) \geq 0$

$h'(0) = -2(f-u, v-u) \geq 0 \Rightarrow *$

Let $u \in K$ be such that $*$

$(f-u, v-u)_H \leq 0 \quad \forall v \in K$

take another $v \in K$

$\| u-f \|^2 - \| v-f \|^2 = \| u \|^2 - \| v \|^2 + 2(v-u, f) =$

$= \| u \|^2 - \| v \|^2 + 2(v-u, f-u) + 2 \underbrace{(v-u, u)}_{(v,u) - \| u \|^2}$

$= 2(v,u) - \| u \|^2 - \| v \|^2 + 2(v-u, f-u)$

$= -\| u-v \|^2 + 2 \underbrace{(v-u, f-u)}_{\leq 0} \leq -\| u-v \|^2$

$\| u-f \|^2 - \| v-f \|^2 \leq -\| u-v \|^2 \quad \forall v \in K$

This proves that u is the unique point of minimum

If K is a closed vector space

$(f-u, v-u) = 0 \quad \forall v \in K$

~~If w~~ First of all we know that

$(f-u, v-u) \leq 0 \quad \forall v \in K$

If $\exists v_0$ s.t.

$(f-u, v_0-u) < 0$

We find a $w \in K$ s.t. $w - u = -(v_0 - u)$

$w = u - v_0 + u = 2u - v_0 \in K$

Prop $\forall\, K \subseteq H$ closed convex

$\exists\ P_K : H \longrightarrow K$

$f \longrightarrow P_K f$ — the point u of the previous Theorem.

It is a contraction:

$$\|P_K f - P_K g\| \leq \|f - g\|$$

Pf