5 Dicembre

Metodo della somiglianza

$$y'' - 3y' + 2y = e^{7x} \quad (1)$$

$$\underbrace{y'' - 3y' + 2y}_{L[y]}$$

$y_g = y_h + y_p$ $\quad$ y_p una soluzione particolare di (1), y_h la soluzione generale di

$$L[y] = 0 \quad (2)$$

(l'omogenea associata)

y_h si ottiene $p(r) = r^2 - 3r + 2 = (r-1)(r-2) = 0$

$r = 1, 2.$ $\quad$ e^x, e^{2x}

$\lambda e^x + \mu e^{2x} = 0$ in un intervallo con $\lambda, \mu \in \mathbb{C}$

$$\Longrightarrow \lambda = \mu = 0$$

$y_h = A e^x + B e^{2y}$ $\quad$ $A, B \in \mathbb{C}$, siccome 7 non e' una radice del polinomio caratteristico, cerchiamo

$$y_p = C e^{7x}$$

$$L[y_p] = L[C e^{7x}] = p(7)\, C e^{7x} = e^{7x}$$

$$p(7)\, C = 1 \qquad C = \frac{1}{p(7)} = \frac{1}{30}$$

$$p(7) = (7-1)(7-2) = 6 \cdot 5 = 30$$

$$y_p = \frac{1}{30} e^{7x}$$

$$L[y] = y'' - 3y' + 2y = e^{2x}$$

$$y_p = u\, e^{2x}$$

$$L[u e^{2x}] = u'' e^{2x} + \underbrace{p'(2)}_{1} u' e^{2x} + \underbrace{p(2)}_{0} u e^{2x} = e^{2x}$$

$$p(r) = (r-1)(r-2) \Rightarrow p(2) = 0$$

$$p'(2) = (r-1)'\underbrace{(r-2)}_{0} + (r-1)\underbrace{(r-2)'}_{1}\Big|_{r=2} = (r-1)\Big|_{r=2} =$$

$$= 1$$

$$= u'' e^{2x} + u' e^{2x} = e^{2x}$$

per

$$u = Cx \qquad u'' = 0 \qquad u' = C$$

$$= C \cancel{e^{2x}} = \cancel{e^{2x}} \qquad C = 1$$

$$y_p = x\, e^{2x}$$

$L[y] = y'' - 3y' + 2y = 6e^x$

$y_p = \underbrace{Cx}_{u} e^x$

$L[Cxe^x] = C\,L[xe^x] =$

$= C\Big(\underbrace{(x)''}_{0} e^x + \underbrace{p'(1)}_{-1}\underbrace{(x)'}_{1} e^x + \underbrace{p(1)}_{0} e^x\Big) = 6e^x$

$p(1) = (r-1)(r-2)\big|_{r=1} = 0$

$p'(1) = (r^2-3r+2)'\big|_{r=1} = 2r-3\big|_{r=1} = -1$

$-C\,\cancel{e^x} = 6\,\cancel{e^x}$ $\qquad C = -6$

$y_p = -6e^x$

$L[y] = y'' - 3y' + 2y = e^{7x} + e^{2x} + 6e^x$

$L[y_1] = e^{7x}$ $\qquad y_1 = \frac{1}{30}e^{7x}$

$L[y_2] = e^{2x}$ $\qquad y_2 = xe^{2x}$

$L[y_3] = 6e^x$. $\qquad y_3 = -6e^x$

$y_p = y_1 + y_2 + y_3$

$L[y_p] = L[y_1 + y_2 + y_3] = \underbrace{L[y_1]}_{e^{7x}} + \underbrace{L[y_2]}_{e^{2x}} + \underbrace{L[y_3]}_{6e^x}$

$= e^{7x} + e^{2x} + 6e^x$

$L[u] = f(x)$

$L[v] = f$

v noto

$u = ?$

$L[u-v] = L[u] - L[v]$

$= f - f$

$= 0$

$u = v + \underbrace{(u-v)}_{\text{è una soluzione della omogenea}}$

$$L[y] = p(x)\, e^{\alpha x}$$

$$L[y] = y'' + b y' + c y$$

$$p(r) = r^2 + br + c$$

Allora, se $p(\alpha) \neq 0$ esiste una $y_p = q(x)\, e^{\alpha x}$ con $\deg q(x) \leq \deg P(x)$

$$L[y] = y'' - 3y' + 2y = (x^6 + x^5)\, e^{3x}$$

$$p(r) \big|_{r=3} = (r-1)(r-2) \big|_{r=3} = 2 \neq 0$$

$$y_p = (\underbrace{a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}_{u(x)})\, e^{3x}$$

$$L[y_p] = L[u\, e^{3x}] =$$

$$= \Big(u'' + u' \overset{3}{p'(3)} + u \overset{2}{p(3)}\Big) \cancel{e^{3x}} = (x^6 + x^5) \cancel{e^{3x}}$$

$$p'(r) \big|_{r=3} = 2r - 3 \big|_{r=3} = 3$$

$$6\cdot 5\, a_6 x^4 + 5\cdot 4\, a_5 x^3 + 4\cdot 3\, a_4 x^2 + 3\cdot 2\, a_3 x + 2a_2 +$$

$$+3(6 a_6 x^5 + 5 a_5 x^4 + 4 a_4 x^3 + 3 a_3 x^2 + 2 a_2 x + a_1)$$

$$+2(a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0) = x^6 + x^5$$

$$\boxed{\begin{aligned} &2a_6 x^6 + x^5(2a_5 + 18a_6) + x^4(2a_4 + 15a_5 + 30a_6) \\ &+ x^3(2a_3 + 12a_4 + 20a_5) + x^2(2a_2 + 9a_3 + 12a_4) \\ &+ x(2a_1 + 6a_2 + 6a_3) + 2a_0 + 3a_1 + 2a_2 = x^6 + x^5 \end{aligned}}$$

$$\begin{cases} 2a_6 = 1 \\ 2a_5 + 18a_6 = 1 \\ 2a_4 + 15a_5 + 30a_6 = 0 \\ 2a_3 + 12a_4 + 20a_5 = 0 \\ 2a_2 + 9a_3 + 12a_4 = 0 \\ 2a_1 + 6a_2 + 6a_3 = 0 \\ 2a_0 + 3a_1 + 2a_2 = 0 \end{cases}$$

$$a_6 = \frac{1}{2}$$

$$a_5 = \frac{1 - 18a_6}{2} = \frac{1-9}{2} = -\frac{8}{2} = -4$$

$$L[y] = y'' - 3y' + 2y = x^2 e^{1x}$$

$$p(r) = (r-2)(r-1) = r^2 - 3r + 2$$

$$y_p = x\,(ax^2 + bx + c)\,e^x$$

$$= (ax^3 + bx^2 + cx + d)\,e^x$$

$$L[y_p] = L[ax^3 e^x + b x^2 e^x + c x e^x + d e^x]$$

$$= L[(ax^3 + bx^2 + cx)e^x] + \underbrace{L[de^x]}_{d\,L[e^x]} = 0$$

$$= x^2 e^x \qquad\qquad d\,\underbrace{p(1)}_{0}\,e^x$$

$$y_p = u\,e^x \qquad u = x(ax^2 + bx + c) = ax^3 + bx^2 + cx$$

$$L[y_p] = (u'' + \overset{-1}{p'(1)}\,u')\,\cancel{e^x} = \cancel{e^x}\,x^2$$

$$u' = 3ax^2 + 2bx + c$$

$$u'' = 6ax + 2b$$

$$u'' - u' = 6ax + 2b - 3ax^2 - 2bx - c = x^2$$

$$-3ax^2 + x(6a - 2b) + 2b - c = x^2$$

$$\begin{cases} -3a = 1 \\ 6a - 2b = 0 \\ 2b - c = 0 \end{cases}$$

$$\underbrace{y'' - 2y' + y}_{L[y]} = x^2 e^x$$

$$p(r)\Big|_{r=1} = r^2 - 2r + \underbrace{1}_{r=1} = (r-1)^{2}\Big|_{r=1} = 0$$

$$p'(r)\Big|_{r=1} = 2(r-1)\Big|_{r=1} = 0$$

$$\Bigg(\quad p(r) = (r-1)^4 \, r$$

$$p(1) = p'(1) = p''(1) = p'''(1) = 0$$

$$p'(r) = 4(r-1)^3 r + (r-1)^4 = 0 \quad \text{por } r = 1$$

$$p''(r) = 12(r-1)^2 r + 8(r-1)^3 = 0 \quad \text{por } r = 1$$

$$p'''(r) = 24(r-1) r + (12 + 24)(r-1)^2 = 0 \quad \text{por } r = 1$$

$$p^{(4)}(r) = 24 r + (r-1)\left(2\cancel{4} + 2(12+24)\right)\Big|_{r=1} = 24 \quad \Bigg)$$

$$\underbrace{y'' - 2y' + y}_{L[y]} = x^2 e^x$$

$$p(r)\Big|_{r=1} = r^2 - 2r + 1\Big|_{r=1} = (r-1)^2\Big|_{r=1} = 0$$

$$p'(r)\Big|_{r=1} = 2(r-1)\Big|_{r=1} = 0 \qquad L[xe^x] = 0$$

$$L[e^x] = 0$$

$$y_p = u\, e^x \qquad u = x^2(ax^2 + bx + c) = = ax^4 + bx^3 + cx^2$$

$$L[u\, e^x] =$$

$$= [u'' + u' \overset{0}{p'(1)} + u \overset{0}{p(1)}]\, \cancel{e^x} = x^2 \cancel{e^x}$$

$$u'' = x^2$$

$$u' = 4ax^3 + 3bx^2 + 2cx$$

$$u'' = 12ax^2 + 6bx + 2c$$

$$u'' = x^2 \qquad 12ax^2 + 6bx + 2c = x^2$$

$$\begin{cases} 12a = 1 \\ 6b = 0 \\ 2c = 0 \end{cases}$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = \varphi(x) \quad e^{\alpha x}$$

$$p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$$

se $p(\alpha) \neq 0$ allora si cerca una soluzione

$y_p = u(x)\, e^{\alpha x}$ $u(x)$ un polinomio $\deg u = \deg \varphi$

se invece α è una radice di $p(r)$ di molteplicità m

$y_p = x^m u(x)\, e^{\alpha x}$ dove

$$y'' + y = \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$p(r) = r^2 + 1 \qquad p(\pm i) = 0$$

$$y_p = A\,x\cos x + B\,x\sin x$$

$$L[y_p] = A\,L[x\cos x] + B\,L[x\sin x] = \sin x$$

$$L[x\cos x] = (x\cos x)'' + x\cos x = (\cos x - x\sin x)' + x\cos x$$

$$= -2\sin x - \cancel{x\cos x} \underbrace{- x\cos x + x\cos x}_{0}$$

$$= -2\sin x + \cancel{x\cos x}$$

$$L[x\sin x] = (x\sin x)'' + x\sin x = (\sin x + x\cos x)' + x\sin x$$

$$= 2\cos x - \cancel{x\sin x} + \cancel{x\sin x}$$

$$L[y_p] = A\,\underbrace{L[x\cos x]}_{-2\sin x} + B\,\underbrace{L[x\sin x]}_{2\cos x}$$

$$= -2A\sin x + 2B\cos x = \sin x + 0\cos x$$

$$-2A = 1 \qquad A = -\frac{1}{2}$$

$$2B = 0$$

$$y_p = -\frac{x}{2}\cos x$$

$$y_g = y_h + y_p = \boxed{C_1\cos x + C_2\sin x - \frac{x}{2}\cos x}$$

$$y'' + y = \sin x$$

$$y'' + y = \sin 2x \qquad p(\pm 2i) \neq 0$$

$$y_p = \underline{A\cos(2x) + B\sin(2x)}$$

x_0

$$\begin{cases} y'' + by' + cy = f(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

problema dei dati iniziali di Cauchy

$$\begin{cases} y'' + y = \sin x \\ y(0) = y_0 = 1 \\ y'(0) = y_1 = 2 \end{cases}$$

$$y = A\cos x + B\sin x - \frac{x}{2}\cos x$$

$$y(0) = A = 1$$

$$y'(0) = B - \frac{1}{2} = 2$$

$$\begin{cases} A = 1 \\ B = \frac{5}{2} \end{cases}$$