Lezione 39 - 5/12/2025

$f(x) = \frac{1}{x} + 2\,\text{arctg}\, x$

$\text{arctg}\, 1 = \frac{\pi}{4}$

$\text{arctg}(-1) = -\frac{\pi}{2}$

$D = \mathbb{R} \setminus \{0\}$

segno $\quad f(x) > 0$ per $x > 0$, $\quad f(x) < 0$ per $x < 0$

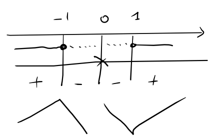
limiti $\quad \lim_{x \to +\infty} f(x) = \pi$, $\quad \lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to 0^-} f(x) = -\infty$, $\quad \lim_{x \to -\infty} f(x) = -\pi$

$f'(x) = -\frac{1}{x^2} + \frac{2}{1+x^2} = \frac{x^2-1}{x^2(1+x^2)}$

segno di $f'(x)$ $\quad N \geq 0$ per $x^2 - 1 \geq 0$ cioè per $x \leq -1 \cup x \geq 1$

$D > 0$ per $x \neq 0$



massimo per $x = -1$,

$f(-1) = -1 - 2\frac{\pi}{4} = -1 - \frac{\pi}{2}$

minimo per $x = 1$

$f(1) = 1 + \frac{\pi}{2}$

$2x^3 + 2x + 2x^3 = 4x^3 + 2x$

$$f''(x) = \frac{2x\left(x^2(x^2+1)\right) - \left(2x(x^2+1) + 2x \cdot x^2\right)(x^2-1)}{\left(x^2(x^2+1)\right)^2}$$

$$= \frac{2x^5 + \cancel{2x^3} - 4x^5 - \cancel{2x^3} + 4x^3 + 2x}{(x^2(x^2+1))^2} = \frac{-2x^5 + 4x^3 + 2x}{(x^2(x^2+1))^2}$$

studio il segno di $-2x^5 + 4x^3 + 2x$

$-2x(x^4 - 2x^2 - 1)$

studio il segno di $x^4 - 2x^2 - 1$

pongo $x^2 = t$ $\quad t^2 - 2t - 1 > 0$

$t_{1,2} = 1 \mp \sqrt{1+1} = \begin{matrix} 1+\sqrt{2} \\ 1-\sqrt{2} \end{matrix}$

$t^2 - 2t - 1 \geq 0$ per

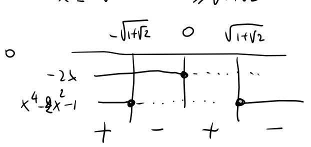
$t \leq 1 - \sqrt{2} \cup t \geq 1 + \sqrt{2}$

$x^4 - 2x^2 - 1 \geq 0$ per $\underbrace{x^2 \leq 1 - \sqrt{2}}_{\emptyset} \vee x^2 \geq 1 + \sqrt{2}$

$x \leq -\sqrt{1+\sqrt{2}} \vee$
$x \geq \sqrt{1+\sqrt{2}}$

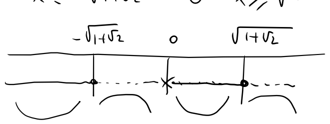
riassunto $x^4 - 2x^2 - 1 \geq 0$ per $x \leq -\sqrt{1+\sqrt{2}} \cup x \geq \sqrt{1+\sqrt{2}}$

$-2x(x^4 - 2x^2 - 1) \geq 0$



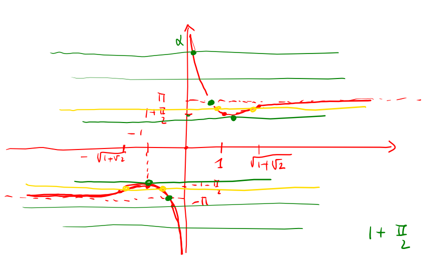
$$f''(x) = \frac{-2x(x^4 - 2x^2 - 1)}{(x^2(x^2+1))^2}$$

$f''(x) \geq 0$ per $x \leq -\sqrt{1+\sqrt{2}} \quad \cup \quad x \geq \sqrt{1+\sqrt{2}}$



flessi in $x = -\sqrt{1+\sqrt{2}}$ e $x = \sqrt{1+\sqrt{2}}$

grafico



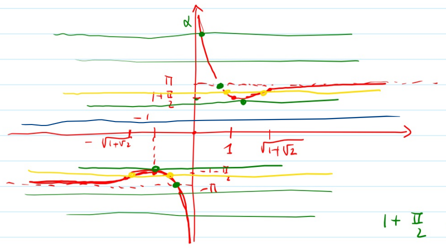
al variare di α quante soluzioni

$\alpha = \frac{1}{x} + 2\,\text{arctg}\, x$

$\alpha \geq \pi \quad \vee \quad \alpha \leq -\pi$
1 sol.

$1 + \frac{\pi}{2} < \alpha < \pi \quad \vee \quad -\pi < \alpha < -1 - \frac{\pi}{2}$
2 sol.

grafico



al variare di α quante soluzioni

$\alpha = \frac{1}{x} + 2\,\mathrm{arctg}\, x$

$\alpha \geq \pi \ \vee \ \alpha \leq -\pi$ 1 sol.

$1+\frac{\pi}{2} < \alpha < \pi \ \vee \ -\pi < \alpha < -1-\frac{\pi}{2}$ 2 sol.

$\alpha = 1+\frac{\pi}{2} \ \vee \ \alpha = -1-\frac{\pi}{2}$ 1 sol.

$-1-\frac{\pi}{2} < \alpha < 1+\frac{\pi}{2}$ nessuna sol.

INTEGRALE DI RIEMANN

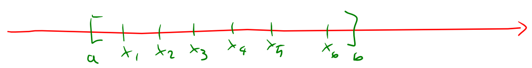
sia $[a,b]$ intervallo chiuso e limitato $(a, b \in \mathbb{R})$

def. chiamo suddivisione di $[a,b]$

un insieme finito di punti di $[a,b]$, che contenga sia il punto a che il punto b

lo indico con $\Delta = \{a, b, x_1, x_2, \ldots, x_{n-1}\}$

con $\underset{x_0}{a} < x_1 < x_2 < \ldots < x_{n-1} < \underset{x_n}{b}$



$\{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b\} = \Delta$

Indico con $\mathcal{D}$ l'insieme delle suddivisioni di $[a,b]$

def sia $f : [a,b] \to \mathbb{R}$, f sia limitata

(significa $\exists M > 0 : \ \forall x \in [a,b], \ |f(x)| \leq M$)

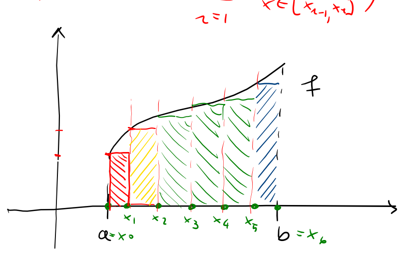
sia $\Delta \in \mathcal{D}$ una suddivisione di $[a,b]$

chiamo somma inferiore per f relativamente a Δ

il numero $$s(f,\Delta) = \sum_{i=1}^{n} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

somma superiore (per f rel. a Δ)

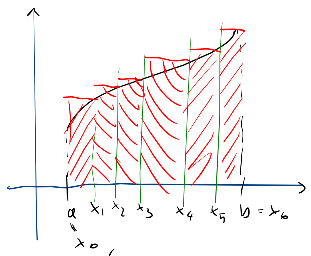
$$S(f,\Delta) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) \cdot (x_i - x_{i-1})$$



$\Delta = \{\overset{a}{x_0}, x_1, x_2, x_3, x_4, x_5, \overset{b}{x_6}\}$

$$s(f,\Delta) = \sum_{i=1}^{6} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right)(x_i - x_{i-1}) = \underbrace{\left(\inf_{x \in [a, x_1]} f \right)(x_1 - a)}_{\text{area rettangolo}} + \underbrace{\left(\inf_{x \in [x_1, x_2]} f(x) \right)(x_2 - x_1)}_{\text{area}} + \ldots + \underbrace{\left(\inf_{x \in [x_{n-1}, b]} f(x) \right)(b - x_{n-1})}_{\text{area}}$$

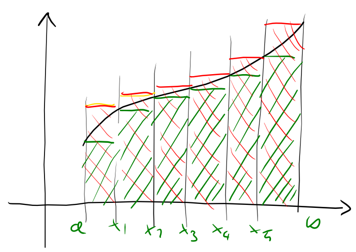
$s(f,\Delta)$ = area plurirettangolo sotto il grafico



$$S(f,\Delta)=\sum_{i=1}^{6}\left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)(x_i-x_{i-1})$$

$$=\left(\sup_{x\in[a,x_1]} f(x)\right)(x_1-a)+\left(\sup_{x\in[x_1,x_2]} f(x)\right)(x_2-x_1)+\dots+\left(\sup_{x\in[x_{n-1},b]} f(x)\right)(b-x_{n-1})$$

$S(f,\Delta)$ = area plurirettangolo sopra il grafico



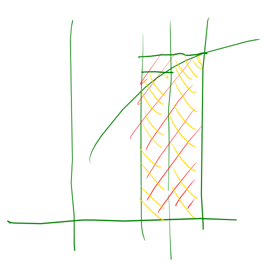
Proprietà delle suddivisioni

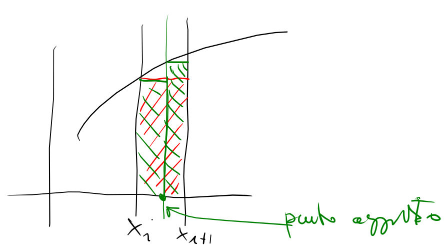
1) $s(f,\Delta) \leq S(f,\Delta)$

perché $\inf_{[x_{i-1},x_i]} f \leq \sup_{[x_{i-1},x_i]} f$

2) supponiamo $\Delta_1 \subseteq \Delta_2$ (Δ_2 ha i punti di Δ_1 più altri punti)

$$s(f,\Delta_1) \leq s(f,\Delta_2)$$





$$S(f,\Delta_1) \geq S(f,\Delta_2)$$

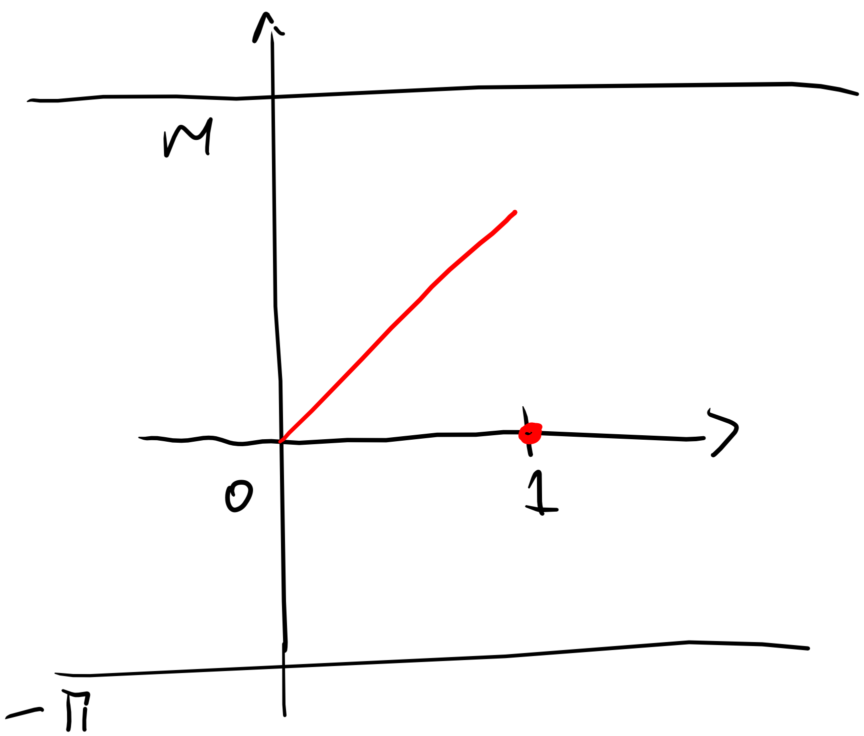
- Δ_1, Δ_2 suddivisioni $\Rightarrow$ $\Delta_1 \cup \Delta_2$ è suddivisione

$$s(f,\Delta_1) \leq s(f,\Delta_1\cup\Delta_2)$$
$$S(f,\Delta_2) \geq S(f,\Delta_1\cup\Delta_2)$$

$$\Rightarrow \quad s(f,\Delta_1) \leq s(f,\Delta_1\cup\Delta_2) \leq S(f,\Delta_1\cup\Delta_2) \leq S(f,\Delta_2)$$

$f : [0,1] \to \mathbb{R}$

limitata ma senza massimo.



$$f(x) = \begin{cases} x & \text{se } 0 \leq x < 1 \\ 0 & \text{se } x = 0 \end{cases}$$

$f([0,1]) = [0,1[$

$1 = \sup f$

1 non è il max f

f non ha max

l'insieme immagine non ha max ma solo sup