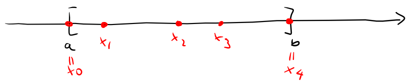
Lezione 40 - 9/12/2025

Integrale di Riemann

def. dato $[a,b]$ int. chiuso e limitato

diciamo <u>suddivisione</u> di $[a,b]$ un insieme finito di pti di $[a,b]$, contenenti a e b

$$\Delta = \{ a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b \}$$



$\mathcal{D}$ = insieme delle suddivisioni

def. sia $f : [a,b] \to \mathbb{R}$ f sia limitata

$\Updownarrow$

$\exists M > 0 : \forall x \in [a,b], \ |f(x)| < M$

sia $\Delta \in \mathcal{D}$, Δ una suddivisione $(\Delta = \{a < x_1 < x_2 \dots < x_{n-1} < b\})$

diciamo <u>somma inferiore</u> per f relativamente a Δ

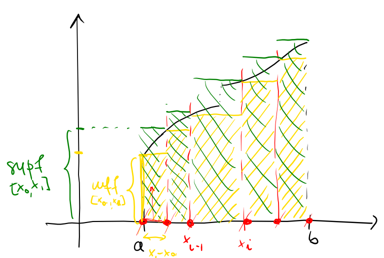
$$s(f,\Delta) = \sum_{i=1}^{n} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

l'est. inferiore di f su $[x_{i-1}, x_i]$

lunghezza dell' int. $[x_{i-1}, x_i]$

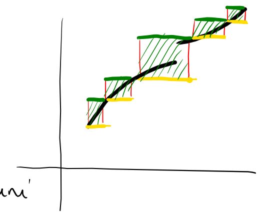
<u>somma superiore</u> per f rel. a Δ

$$S(f,\Delta) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) \cdot (x_i - x_{i-1})$$



$s(f,\Delta)$ = area (gialla)

$S(f,\Delta)$ = area (verde)



Proprietà delle somme superiori e inferiori

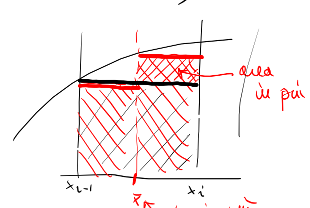
- $S(f,\Delta) \geq s(f,\Delta)$ (perché $\sup_{[x_{i-1}, x_i]} f \geq \inf_{[x_{i-1}, x_i]} f$

- Siano Δ_1, Δ_2 siano 2 suddivisioni

  $\Delta_1 \subseteq \Delta_2$ (si potrebbe dire che Δ_2 è "più fine" di Δ_1)

  $s(f,\Delta_1) \leq s(f,\Delta_2)$

  $S(f,\Delta_1) \quad S(f,\Delta_2)$



- Siano Δ_1, Δ_2 siano 2 suddivisioni

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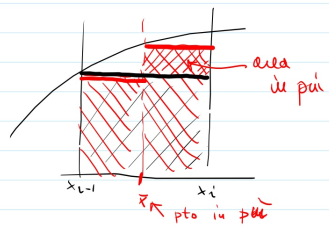
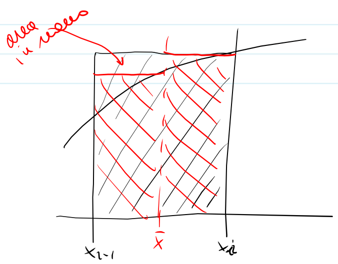
$s(f,\Delta_1) \leq s(f,\Delta_2)$

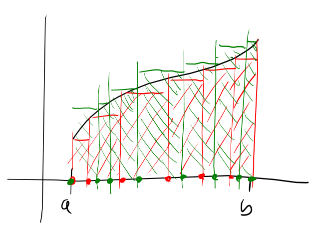
$S(f,\Delta_1) \geq S(f,\Delta_2)$





- siano Δ_1 e Δ_2 2 suddivisioni (qualunque)

$$s(f,\Delta_1) \leq S(f,\Delta_2)$$



$$s(f,\Delta_1) \leq s(f,\Delta_1 \cup \Delta_2) \leq S(f,\Delta_1 \cup \Delta_2) \leq S(f,\Delta_2)$$

Conclusione

$f : [a,b] \to \mathbb{R}$, limitata

$$\sup_{\Delta \in \mathcal{D}} \{ s(f,\Delta) \} \leq \inf_{\Delta \in \mathcal{D}} \{ S(f,\Delta) \}$$

def. $f : [a,b] \to \mathbb{R}$, f limitata

se $\sup_{\Delta \in \mathcal{D}} \{ s(f,\Delta) \} = \inf_{\Delta \in \mathcal{D}} \{ S(f,\Delta) \}$

f si dice integrabile secondo Riemann su $[a,b]$

si scrive $f \in \mathcal{R}([a,b])$

e il valore $\sup_{\Delta \in \mathcal{D}} \{ s(f,\Delta) \} = \inf_{\Delta \in \Delta} \{ S(f,\Delta) \} = \underbrace{\int_a^b f(x)\,dx}_{\text{integrale tra a e b di } f} = \int_{[a,b]} f$

Condizione necessaria e sufficiente per l'integrabilità secondo Riemann.

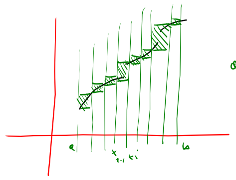
Condizione necessaria e sufficiente per l'integrabilità secondo Riemann.

Teorema: sia $f:[a,b]\to\mathbb{R}$, limitata

f è integrabile s. Riemann

se e solo se

$$\forall \varepsilon>0,\ \exists \Delta\in\mathcal{D}: \quad S(f,\Delta)-s(f,\Delta)<\varepsilon \quad (*)$$



area ▨ $<\varepsilon$

dim. f è int. s. Riemann $\overset{?}{\Longrightarrow}$ $(*)$

$$\sup\{s(f,\Delta)\}=\inf\{S(f,\Delta)\}=\int_a^b f(x)\,dx\in\mathbb{R}$$

2° prop sup: fisso ε, $\exists \Delta_1$: $s(f,\Delta_1)>\int_a^b f(x)\,dx-\varepsilon/2$

2° prop inf: fisso ε, $\exists \Delta_2$: $S(f,\Delta_2)<\int_a^b f(x)\,dx+\varepsilon/2$

se considero $\Delta_1\cup\Delta_2=\tilde\Delta$

$$s(f,\tilde\Delta)>\int_a^b f(x)\,dx-\varepsilon/2$$
$$S(f,\tilde\Delta)<\int_a^b f(x)\,dx+\varepsilon/2$$

$$S(f,\tilde\Delta)-s(f,\tilde\Delta)<\underbrace{\int_a^b f(x)\,dx+\varepsilon/2-\left(\int_a^b f(x)\,dx-\varepsilon/2\right)}_{=\varepsilon}$$

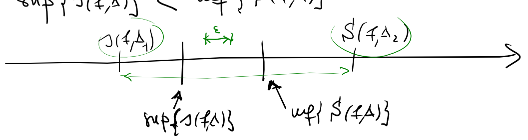
ho $(*)$

Viceversa $(*)\overset{?}{\Longrightarrow}$ int. s. Riemann

invece provo non int. s. Riemann $\Rightarrow$ non $(*)$

non int. s. Riemann

$$\sup\{s(f,\Delta)\}<\inf\{S(f,\Delta)\}$$



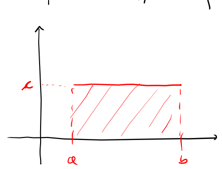
è sufficiente scegliere $\varepsilon=\dfrac{\inf\{S\}-\sup\{s\}}{2}$

con questo ε la condizione $(*)$ non vale.

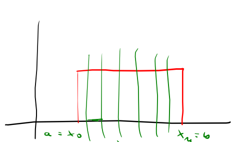
CVD

Esempi.

1) $f:[a,b]\to\mathbb{R}$, $f(x)=c$ (costante)



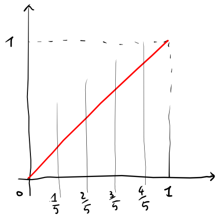
Area $=(b-a)c$



$$s(f,\Delta)=\sum_i \left(\inf_{[x_{i-1},x_i]} f\right)(x_i-x_{i-1}) = \sum_i c(x_i-x_{i-1}) = c\left(\sum_{i=1}^n (x_i-x_{i-1})\right)=c(b-a)$$

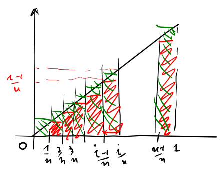
$$S(f,\Delta)=(b-a)c \quad\Longrightarrow\quad \int_a^b f(x)\,dx=(b-a)\cdot c$$

2) $f:[0,1]\to\mathbb{R}$, $f(x)=x$



per provare che è int. s. R.
fisso ε e cerco Δ t.c.
$S(f,\Delta)-s(f,\Delta)<\varepsilon$

prendo Δ t.c. $x_i-x_{i-1}=\frac1n \quad \forall i$



$\sup = \frac{i}{n}$

$$s(f,\Delta)=\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)\cdot\frac1n$$

$\inf\limits_{x\in[x_{i-1},x_i]} f(x)$

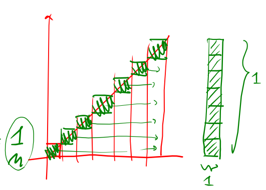
$$=\sum_{i=1}^{n}\frac{(i-1)}{n^2}=\frac{1}{n^2}\sum_{i=1}^{n}(i-1)$$
$$=\frac{1}{n^2}\cdot\sum_{i=0}^{n-1} i$$
$$=\frac{(n-1)n}{2n^2}=\frac{n^2-n}{2n^2}$$

$0+1+2+\dots+n=\frac{n(n+1)}{2}$
$0+1+\quad+(n-1)=\frac{(n-1)\cdot n}{2}$

$$s(f,\Delta)=\frac{n^2-n}{2n^2}=\frac12-\frac1{2n}$$

$$S(f,\Delta)=\frac{n^2+n}{2n^2}=\frac12+\frac1{2n}$$

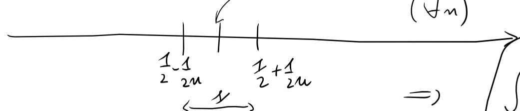
$$S(f,\Delta)-s(f,\Delta)=\frac{n^2+n}{2n^2}-\frac{n^2-n}{2n^2}=\frac{2n}{2n^2}=\frac1n$$



se scelgo $\frac1n<\varepsilon$ ho che $S(f,\Delta)-s(f,\Delta)<\varepsilon$
$\Rightarrow$ int. s. Riemann

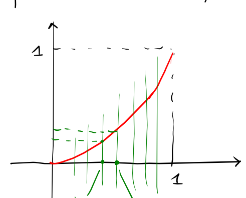
e quanto vale $\int_a^b f(x)dx$?

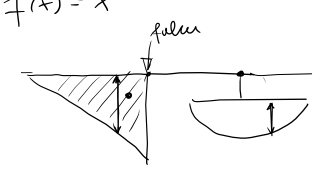
c'è un numero? ($\forall n$) sì è $\frac12$



$\Rightarrow \boxed{\int_0^1 x\,dx=\frac12}$

3) $f:[0,1]\to\mathbb{R}$, $f(x)=x^2$





$$\forall\varepsilon,\exists\Delta,\quad S(f,\Delta)-s(f,\Delta)\overset{?}{<}\varepsilon$$

Δ t.c. $x_i-x_{i-1}=\frac1n$

$$s(f,\Delta)=\sum_{i=1}^{n}\Big(\inf_{(x_{i-1},x_i]} f\Big)\frac1n=\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\cdot\frac1n=\frac1{n^3}\sum_{i=1}^{n}(i-1)^2$$

$$S(f,\Delta)=\sum_{i=1}^{n}\left(\frac in\right)^2\cdot\frac1n=\frac1{n^3}\sum_{i=1}^{n}i^2$$

$0+1+4+9+\dots+n^2=\frac{n(n+1)(2n+3)}{6}$ $\qquad S(f,\Delta)=\frac{n(n+1)(2n+3)}{6n^3}$

$0+1+\dots+(n-1)^2=\frac{(n-1)(n)(2n+1)}{6}$ $\qquad s(f,\Delta)=\frac{(n-1)n(2n+1)}{6n^3}$

$$0+1+4+9+\dots+n^2 = \frac{n(n+1)(2n+3)}{6} \qquad S'(f,\Delta) = \frac{n(n+1)(2n+3)}{6n^3}$$

$$0+1+\dots+(n-1)^2 = \frac{(n-1)(n)(2n+1)}{6} \qquad s(f,\Delta) = \frac{(n-1)n(2n+1)}{6n^3}$$

$$S'(f,\Delta) - s(f,\Delta) =$$

$$(n^2+n)(2n+3) = 2n^3+3n^2+2n^2+3n = 2n^3+5n^2+3n$$
$$(n^2-n)(2n+1) = 2n^3+n^2-2n^2-n = 2n^3-n^2-n$$

$$S'(f,\Delta) - s(f,\Delta) = \frac{6n^2+4n}{6n^3} = \frac{1}{n} + \frac{2}{n^2} < \varepsilon$$

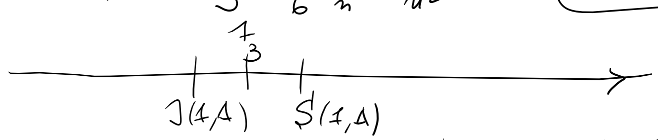
per n abb. grande

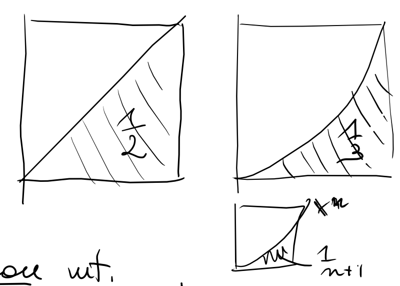
$\Rightarrow$ x^2 è int. s. Riemann

$$S'(f,\Delta) = \frac{1}{3} + \frac{5}{6}\cdot\frac{1}{n} + 3\,\frac{1}{n^2}$$

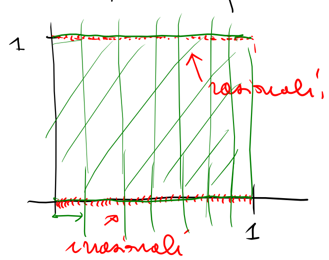
$$s(f,\Delta) = \frac{1}{3} - \frac{1}{6}\cdot\frac{1}{n} - \frac{1}{n^2}$$

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$





4) un esempio di funzione non int. s. Riemann



$$f(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ 0 & \text{se } x \notin \mathbb{Q} \end{cases}$$

$s(f,\Delta) = 0 \quad \forall \Delta$

$S'(f,\Delta) = 1 \quad \forall \Delta$

DUE FAMIGLIE DI FUNZIONI INT. S. RIEMANN

Teo. sia $f:[a,b]\to\mathbb{R}$, f sia monotona. Allora f è int. S. Riemann.

Teorema sia $f:[a,b]\to\mathbb{R}$, f continua. Allora f è int. s. Riemann.