Lezione 41 - 11/12/2025

Integrale di Riemann.

- $[a,b]$ intervallo (chiuso e limitato) $\longrightarrow$ suddivisione
  $\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$

- $f : [a,b] \to \mathbb{R}$, limitata

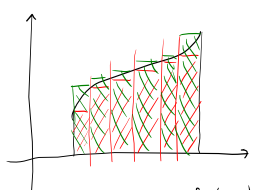
  $$s(f,\Delta) = \sum_{i=1}^{n} \Big(\inf_{x \in [x_{i-1}, x_i]} f(x) \Big)(x_i - x_{i-1})$$

  $$S(f,\Delta) = \sum_{i=1}^{n} \Big(\sup_{x \in [x_{i-1}, x_i]} f(x) \Big)(x_i - x_{i-1})$$

- $\forall \Delta_1, \Delta_2 \in \mathcal{D} = \{\text{suddivisioni}\}$

  $$s(f,\Delta_1) \leq S(f,\Delta_2)$$

- f è int. s. Riemann se $\sup_{\Delta \in \mathcal{D}} \{s(f,\Delta)\} = \inf_{\Delta \in \mathcal{D}} \{S(f,\Delta)\}$



$s(f,\Delta) =$ area

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$$\sup_{\Delta \in \mathcal{D}} \{s(f,\Delta)\} = \inf_{\Delta \in \mathcal{D}} \{S(f,\Delta)\} = \int_a^b f(x)\,dx = \int_{[a,b]} f$$

f non int. s. Riemann

Teo. f è int. s. Riemann $\big(f \in \mathcal{R}([a,b])\big)$

$\Updownarrow$

$\forall \varepsilon, \exists \Delta \in \mathcal{D} : S(f,\Delta) - s(f,\Delta) < \varepsilon$.

Teorema (integrabilità delle monotone)

$f : [a,b] \to \mathbb{R}$, f monotona (per es. crescente)

Allora f è int. s. Riemann.

dim. utilizziamo la caratterizzazione

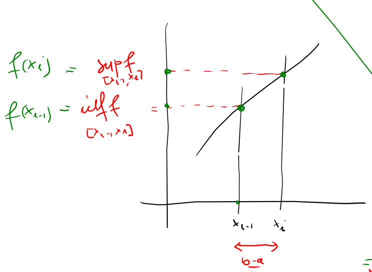
fisso $\varepsilon > 0$. se trovo Δ : $S(f,\Delta) - s(f,\Delta) < \varepsilon$ ho finito

ε sia fissato.

considero Δ t.c. $\forall i = 1, \ldots, n$ $(x_i - x_{i-1}) = \frac{b-a}{n}$

è la suddivisione in cui ci sono n intervallini tutti lunghi uguale,

$$S(f,\Delta) - s(f,\Delta) = \sum_{i=1}^{n} \Big(\underbrace{\sup_{x \in [x_{i-1}, x_i]} f}_{= f(x_i)} - \underbrace{\inf_{x \in [x_{i-1}, x_i]} f}_{= f(x_{i-1})} \Big) \underbrace{(x_i - x_{i-1})}_{= \frac{b-a}{n}}$$



$$= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1}) \big)$$

$$\| $$

$$\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cancel{f(x_3)} - \cancel{f(x_2)} + \ldots + f(x_n) - \cancel{f(x_{n-1})}$$

$$= \frac{b-a}{n} \cdot \big(f(x_n) - f(x_0) \big)$$

$$= \frac{(b-a)}{n} \big(f(b) - f(a) \big)$$

$S(f,\Delta) - s(f,\Delta) = \frac{(b-a)}{n}(f(b)-f(a))$

basta prendere n t.c. $\frac{(b-a)(f(b)-f(a))}{n} < \varepsilon$

C.V.D.

Teo (integrabilità delle funzioni continue)

$f: [a,b] \to \mathbb{R}$, f sia continua

Allora f è int. s. Riemann.

dim. Ricordo il teorema di HEINE

se $f: [a,b] \to \mathbb{R}$ è continua allora f è unif. continua

$\forall \varepsilon > 0, \exists \delta > 0 : \forall x_1, x_2 \in [a,b]$

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

fisso ε e considero δ t.c. se $|x_1 - x_2| < \delta$ allora $|f(x_1) - f(x_2)| < \frac{\varepsilon}{b-a}$

considero $\Delta = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$

tale che $\forall i$, $|x_i - x_{i-1}| < \delta$

e considero $S(f,\Delta) - s(f,\Delta) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right)(x_i - x_{i-1})$

$\sup_{x \in [x_{i-1}, x_i]} f(x) = \max_{x \in [x_{i-1}, x_i]} f(x) = f(x_{max,i})$

$\inf_{x \in [x_{i-1}, x_i]} f(x) = \min_{x \in [x_{i-1}, x_i]} f(x) = f(x_{min,i})$ ← t. di Weie...

$x_{max,i}, x_{min,i} \in [x_{i-1}, x_i]$

$|x_{max,i} - x_{min,i}| < \delta$

$\Downarrow$

$|f(x_{max}) - f(x_{min})| < \varepsilon / b-a$

$$\Rightarrow S(f,\Delta) - s(f,\Delta) \leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1})$$

$$\leq \frac{\varepsilon}{b-a} \cdot \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$\leq \frac{\varepsilon}{\cancel{(b-a)}} \cdot \cancel{(b-a)} = \varepsilon$$

C.V.D.

Altre proprietà dell'integrale

1) se $f, g \in \mathcal{R}([a,b])$ allora $f \pm g \in \mathcal{R}([a,b])$

e $\int_a^b (f \pm g)(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

idea so che $\forall \varepsilon > 0, \exists \Delta_1 :$ $S(f,\Delta_1) - s(f,\Delta_1) < \varepsilon/2$

$\forall \varepsilon > 0, \exists \Delta_2 :$ $S(g,\Delta_2) - s(g,\Delta_2) < \varepsilon/2$

se considero $\tilde{\Delta} = \Delta_1 \cup \Delta_2$

ho $S(f,\tilde{\Delta}) - s(f,\tilde{\Delta}) < \varepsilon/2$

idea: so che $\forall \varepsilon>0,\ \exists \Delta_1 :\quad S(f,\Delta_1) - s(f,\Delta_1) < \varepsilon/2$

$\forall \varepsilon>0,\ \exists \Delta_2 :\quad S(g,\Delta_2) - s(g,\Delta_2) < \varepsilon/2$

se considero $\tilde{\Delta} = \Delta_1 \cup \Delta_2$

ho $S(f,\tilde{\Delta}) - s(f,\tilde{\Delta}) < \varepsilon/2$

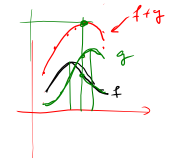
$S(g,\tilde{\Delta}) - s(g,\tilde{\Delta}) < \varepsilon/2$

$$\Big(\underbrace{S(f,\tilde{\Delta})+S(g,\tilde{\Delta})}_{\sum_{i=1}^{n}\left(\sup_{I_i} f + \sup_{I_i} g\right)(x_i-x_{i-1})}\Big) - \Big(s(f,\tilde{\Delta})+s(g,\tilde{\Delta})\Big) < \varepsilon$$

$$s(f,\tilde{\Delta})+s(g,\tilde{\Delta}) = \sum_{i=1}^{n}\left(\inf_{I_i} f + \inf_{I_i} g\right)(x_i-x_{i-1})$$

$\sup_{I_i} (f+g) \leq \sup_{I_i} f + \sup_{I_i} g$

$\inf (f+g) \geq \inf f + \inf g$



$$\Rightarrow \quad S(f+g,\tilde{\Delta}) - s(f+g,\tilde{\Delta}) \leq \underbrace{\left(S(f,\tilde{\Delta})+S(g,\tilde{\Delta})\right) - \left(s(f,\tilde{\Delta})+s(g,\tilde{\Delta})\right)}_{<\ \varepsilon}$$

in conclusione

$$f \pm g \in \mathcal{R}([a,b])$$

e vale $\int_a^b (f\pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

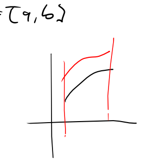
2) se $f \in \mathcal{R}([a,b])$ e $c \in \mathbb{R}$,

allora $cf(x)$ è int. s. Riemann

e $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

3) se $f,g \in \mathcal{R}([a,b])$ e $f(x) \geq g(x)\ \ \forall x \in [a,b]$

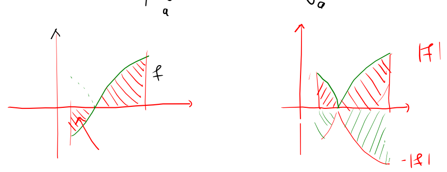
allora $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$



4) se $f \in \mathcal{R}([a,b])$ allora $|f| \in \mathcal{R}([a,b])$

e $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \iff -\int |f| \leq \int f \leq \int |f|$

$-|f|$



5) $f : [a,b] \to \mathbb{R}$ $\quad f \in \mathcal{R}([a,b])$

sia $c \in [a,b]$

Allora $f_{|[a,c]} \in \mathcal{R}([a,c])$

$f_{|[c,b]} \in \mathcal{R}([c,b])$

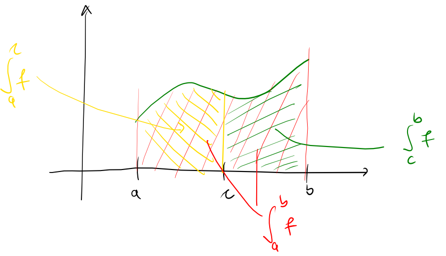
5) $f: [a,b] \to \mathbb{R}$ $\quad f \in \mathcal{R}([a,b])$

sia $c \in [a,b]$

Allora $f_{|[a,c]} \in \mathcal{R}([a,c])$

$f_{|[c,b]} \in \mathcal{R}([c,b])$

e $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

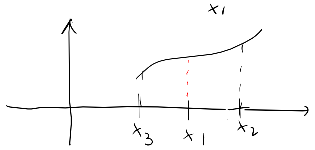


Convenzione: sia $f: [a,b] \to \mathbb{R}$

siano $x_1, x_2, x_3 \in [a,b]$ (non necessariamente $x_1 < x_2 < x_3$)

$$\int_{x_1}^{x_2} f(t)dt = -\int_{x_2}^{x_1} f(t)dt$$

in questo modo $\quad \int_{x_1}^{x_3} f(t)dt = \int_{x_1}^{x_2} f(t)dt + \int_{x_2}^{x_3} f(t)dt$



$\int_{x_1}^{x_3} = -\int_{x_3}^{x_1}$

$\int_{x_1}^{x_2} = \int_{x_1}^{x_2} \quad \Rightarrow \quad \int_{x_1}^{x_2} + \int_{x_2}^{x_3} =$

$\int_{x_2}^{x_3} = -\int_{x_3}^{x_2}$

$\int_{x_1}^{x_2} - \int_{x_3}^{x_2}$

$\int_{x_3}^{x_2} = \int_{x_3}^{x_1} + \int_{x_1}^{x_2}$

$\int_{x_1}^{x_2} - \int_{x_3}^{x_2} = \cancel{\int_{x_1}^{x_2}} - \left(\int_{x_3}^{x_1} + \cancel{\int_{x_1}^{x_2}} \right)$

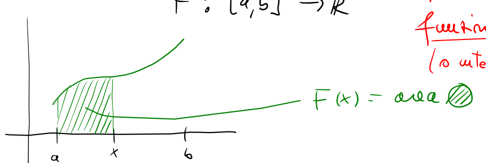
$= -\int_{x_3}^{x_1} = \int_{x_1}^{x_3}$ OK

def. Sia $f: [a,b] \to \mathbb{R}$, f sia int. s. Riemann

definisco $F(x) = \int_a^x f(t)dt$

$F: [a,b] \to \mathbb{R}$

F si dice funzione integrale di f (o integral-funzione di f)



def. Sia $f : [a,b] \to \mathbb{R}$, f sia int. s. Riemann

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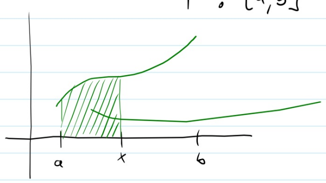


Teorema sia $f : [a,b] \to \mathbb{R}$ int. s. Riemann

Sia $F(x) = \int_a^x f(t)dt$ la sua funzione integrale

Allora F è continua e in particolare vale

$$\exists M>0 : \forall x_1, x_2 \in [a,b], \quad |F(x_2) - F(x_1)| \leq M|x_2 - x_1|$$

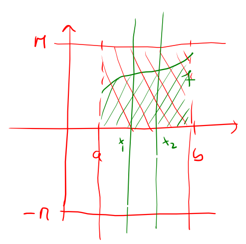
dim. $f \in \mathcal{R}([a,b])$ in particolare f è limitata

cioè $\exists M>0 : \forall x \in [a,b]$

$|f(x)| \leq M$

prendiamo $x_1, x_2 \in [a,b]$ con $x_1 < x_2$

$$\left| \int_{x_1}^{x_2} f(t)dt \right| \leq M|x_1 - x_2|$$



$$\int_{x_1}^{x_2} f = \underset{F(x_2)}{\underset{\parallel}{\int_a^{x_2} f}} - \underset{F(x_1)}{\underset{\parallel}{\int_a^{x_1} f}}$$

$$\int_a^{x_2} = \int_a^{x_1} + \int_{x_1}^{x_2}$$

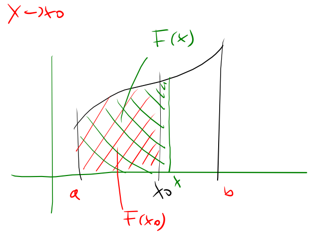
$$\int_{x_1}^{x_2} f = F(x_2) - F(x_1)$$

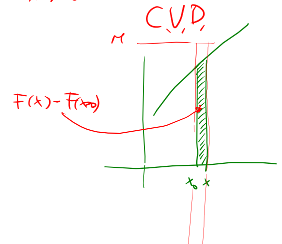
in conclusione $|F(x_2) - F(x_1)| \leq M|x_2 - x_1|$

$$\lim_{x \to x_0} |F(x) - F(x_0)| \leq \lim_{x \to x_0} M|x - x_0| = 0$$

$\Downarrow$ $\lim_{x \to x_0} F(x) - F(x_0) = 0 \Rightarrow \lim_{x \to x_0} F(x) = F(x_0)$

C.V.D.





Teorema (fondamentale del calcolo)

sia $f \in \mathcal{R}([a,b])$

sia $x_0 \in [a,b]$

f sia continua in x_0

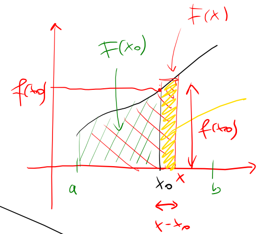
Sia $F : [a,b] \to \mathbb{R}$, $\quad x \longmapsto F(x) = \int_a^x f(t)\,dt$

la sua funzione integrale

Allora F è derivabile in x_0

e $F'(x_0) = f(x_0)$

intuitivamente



$F(x) - F(x_0)$

se considero $\frac{F(x)-F(x_0)}{x-x_0}$

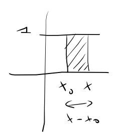
$\lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$

dim.

$F(x) = \int_a^x f(t)\,dt$

$F(x_0) = \int_a^{x_0} f(t)\,dt$

$F(x) - F(x_0) = \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt = \int_{x_0}^x f(t)\,dt$



$R^F_{x_0}(x) = \frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0} \int_{x_0}^x f(t)\,dt$

$f(x_0) = \frac{1}{x-x_0} \cdot f(x_0) \cdot (x - x_0)$ $\qquad x - x_0 = \int_{x_0}^x 1\,dt$

$= \frac{1}{x-x_0} f(x_0) \cdot \int_{x_0}^x 1\,dt$

$= \frac{1}{x-x_0} \cdot \int_{x_0}^x f(x_0)\,dt$

in conclusione $\quad R^F_{x_0}(x) - f(x_0) = \frac{1}{x-x_0} \int_{x_0}^x (f(t) - f(x_0))\,dt$

$= \frac{1}{x-x_0} \int_{x_0}^x f(t)\,dt - \frac{1}{x-x_0} \int_{x_0}^x f(x_0)\,dt$

$\left| R^F_{x_0}(x) - f(x_0) \right| \le \frac{1}{|x-x_0|} \left| \int_{x_0}^x (f(t) - f(x_0))\,dt \right|$

$\le \frac{1}{|x-x_0|} \cdot \left| \int_{x_0}^x |f(t) - f(x_0)|\,dt \right|$

$\lim_{x \to x_0} \left| R^F_{x_0}(x) - f(x_0) \right| \le \lim_{x \to x_0} \frac{\left| \int_{x_0}^x |f(t) - f(x_0)|\,dt \right|}{|x - x_0|}$

se viene 0 ho finito!

so che f è continua in x_0

$\lim_{x \to x_0} f(x) = f(x_0)$

cioè? $\quad \forall \varepsilon > 0, \exists \delta > 0 : \forall x$

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$$\lim_{x\to t_0} \left|R^F_{t_0}(x) - f(t_0)\right| \leq \lim_{x\to t_0} \frac{\left|\int_{t_0}^{x} |f(t)-f(t_0)|\,dt\right|}{|x-t_0|}$$

se viene 0 ho finito!

so che f è continua in x_0

$$\lim_{x\to t_0} f(t) = f(x_0)$$

cioè? $\forall \varepsilon>0, \exists \delta>0 : \forall x$

$$|x-t_0|<\delta \Rightarrow |f(x)-f(t_0)|<\varepsilon$$

allora $|t-t_0|<\delta \Rightarrow |f(t)-f(x_0)|<\varepsilon$

$$\Downarrow$$

$$\left|\int_{t_0}^{x} |f(t)-f(t_0)|\,dt\right| < \varepsilon\,|x-t_0|$$

allora

$$\frac{\left|\int_{t_0}^{x} |f(t)-f(t_0)|\,dt\right|}{|x-t_0|} < \frac{\varepsilon\,\cancel{|x-t_0|}}{\cancel{|x-t_0|}} = \varepsilon$$

in conclusione $\forall \varepsilon>0, \exists \delta>0 : \forall x$

$$|x-t_0|<\delta \Rightarrow \frac{\left|\int_{t_0}^{x} |f(t)-f(t_0)|\,dt\right|}{|x-t_0|} < \varepsilon$$

$$\lim_{x\to t_0} \frac{\left|\int_{t_0}^{x} |f(t)-f(t_0)|\,dt\right|}{|x-t_0|} = 0$$

CVD