11 December

Conv

<u>Def</u> A bounded operator $T: E \to F$ between two B-spaces is compact if it sends bounded sets into relatively compact sets

($\overline{T D_E(0,1)}$ is compact in F).

<u>Example</u> If $\dim R(T) < +\infty$, then we know that $T D_E(0,1)$ is a bounded subset of $R(T)$, $R(T) \subset F$ is closed in F.

and $\overline{T D_E(0,1)}$ is a closed bounded set in $R(T)$ and so it is compact.

(Bolzano Weierstrass).

<u>Remark</u> Let $Tf = \kappa * f$ $\quad \overset{0}{\not\equiv}\ \kappa \in L^q(\mathbb{R}^d)$

$T: L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ $\qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

$\mathbb{R}^d$

Then T is not compact. Here is important the group action of $\mathbb{R}^d$ on the Lebesgue spaces. $\tau_h f = f(\cdot - h)$

Suppose there is a T compact $\quad 1 < p < \infty$

let $f \in L^p(\mathbb{R}^d)$ $\quad f \neq 0$, $\quad h_n \longrightarrow \infty$ in $\mathbb{R}^d$, then $\tau_{h_n} f \xrightarrow{n \to +\infty} 0$

$\{\tau_{h_n} f\}_{n \in \mathbb{N}}$ is a bounded set

$\exists$ a subsequence, which for simplicity I assume equal to the whole sequence, and a $g \in L^r(\mathbb{R}^d)$, s.t. $T\tau_{h_n} f \xrightarrow{n \to +\infty} g$ in $L^r(\mathbb{R}^d)$

$\tau_{h_n} f \longrightarrow 0 \Rightarrow T \tau_{h_n} f \xrightarrow{n \to +\infty} 0$

$\Rightarrow g = 0$

$T\tau_{h_n} f = \kappa * \tau_{h_n} f = \tau_{h_n} \kappa * f =$
$= \tau_{h_n} T f$

And since $\tau_{h_n} T f \longrightarrow 0 = g \Longleftrightarrow$
$\lim_{n \to +\infty} |\tau_{h_n} T f|_{L^r(\mathbb{R}^d)} = 0 \Longleftrightarrow Tf = 0$

But
If $\kappa \neq 0$ $\exists$ $f \in L^p(\mathbb{R}^d)$ s.t. $\underbrace{\kappa * f}_{Tf} \neq 0$

If $p = 2$, $r = 2$, $q = 1$

$\widehat{\kappa * f} = \hat{\kappa} \hat{f}$

(Hardy-Littlewood-Sobolev)

<u>Theorem</u> $0<\gamma<d$ $\quad 1<p<q<\infty$

(1) $\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$

$$Tf(x) = \int_{\mathbb{R}^d} |x-y|^{-\gamma} f(y)\, dy$$

for $f \in C_c^0(\mathbb{R}^d)$ extends into a bounded operator that is $\exists\, C>0$ s.t.

$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \quad \forall f \in L^p(\mathbb{R}^d)$$

<u>Observation</u> 1) These operators are non compact

In fact T is translation invariant

2) $T: L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ $\quad \forall \lambda>0$

$S_{p\lambda} f = \lambda^{\frac{d}{p}} f(\lambda\,\cdot)$ is an isometry in $L^p(\mathbb{R}^d)$

$S_{q\lambda} g = \lambda^{\frac{d}{q}} g(\lambda\,\cdot)$ $\quad$ " $\quad L^q(\mathbb{R}^d)$

$1 = \frac{\lambda^{\gamma}}{\lambda^{\gamma}}$

$$T S_{p\lambda} f = \int_{\mathbb{R}^d} |x-y|^{-\gamma} \lambda^{\frac{d}{p}} f(\lambda y)\, dy =$$

$$= \lambda^{\gamma-d} \lambda^{\frac{d}{p}} \int_{\mathbb{R}^d} |\lambda x - \lambda y|^{-\gamma} f(\lambda y) \lambda^d\, dy$$

$$= \lambda^{\gamma-d} \lambda^{\frac{d}{p}} \int_{\mathbb{R}^d} |\lambda x - y|^{-\gamma} f(y)\, dy$$

$$= \lambda^{\gamma-d} \lambda^{\frac{d}{p}} Tf(\lambda x) = \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} \lambda^{\frac{d}{q}} Tf(\lambda x)$$

$$= \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} S_{q\lambda} Tf$$

$$T S_{p\lambda} f = \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} S_{q\lambda} Tf$$

$* \quad \|Tf\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \quad \forall f \in L^p(\mathbb{R}^d)$

$$\|T S_{p\lambda} f\|_{L^q(\mathbb{R}^d)} \leq C \|S_{p\lambda} f\|_{L^p(\mathbb{R}^d)} = C\|f\|_{L^p(\mathbb{R}^d)}$$

$$\lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} \|S_{q\lambda} Tf\|_{L^q(\mathbb{R}^d)} = \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} \|Tf\|_{L^q(\mathbb{R}^d)} \quad \forall \lambda>0$$

so $* \Rightarrow$

$\forall \lambda>0 \quad \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}} \|Tf\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$

If this is true, necessarily implies $\gamma - d - \frac{d}{q} + \frac{d}{p} = 0$

$\gamma - d - \frac{d}{q} + \frac{d}{p} = 0 \quad (*)$

$(*)$ is same as (1)

In the situation of the Theorem $\lambda^{\overbrace{\gamma-d-\frac{d}{q}+\frac{d}{p}}^{0}}$

$$\boxed{T S_{p\lambda} f = S_{q\lambda} Tf}$$

$\lambda_n \to +\infty$

$S_{p\lambda_n} f \to 0 \Rightarrow T S_{p\lambda_n} f \to 0$ in $L^q(\mathbb{R}^d)$

$$\{T S_{p\lambda_n} f\}_{n\in\mathbb{N}} = \{S_{q\lambda_n} Tf\}_{n\in\mathbb{N}}$$

if T is compact, there exists a convergent subsequence. It is not restrictive to assume

$S_{q\lambda_n} Tf \to g$ in $L^q(\mathbb{R}^d)$

$\Rightarrow g=0 \Rightarrow \|S_{q\lambda_n} Tf\|_{L^q(\mathbb{R}^d)} \to 0$

$\|Tf\|_{L^q(\mathbb{R})} = 0$

so $Tf=0 \quad \forall f \in L^p(\mathbb{R}^d)$, but this is not true

This is useful because if

$$f \in L^p(D_{\mathbb{R}^d}(0,1)) \xrightarrow{T} L^q(D_{\mathbb{R}^d}(0,1))$$

then for $\lambda_n > 1 \quad S_{p\lambda_n} f \in L^p(D_{\mathbb{R}^d}(0,1))$

$\underline{K(E,F)} = \{ T \in \underbrace{\mathcal{L}(E,F)} : T \text{ compact} \}$

Lemma $K(E,F)$ is closed in norm

Pf Let $T \in \overline{K(E,F)}^{\left(\mathcal{L}(E,F), \|\cdot\|_{E \to F}\right)}$

To show that T is compact it is enough to prove that $T D_E(0,1)$ is relatively compact in F.

$\overline{T D_E(0,1)}$ is compact in F

$\iff \overline{T D_E(0,1)}$ is sequentially compact (any sequence inside $\overline{T D_E(0,1)}$ has a subsequence convergente in $\overline{T D_E(0,1)}$)

It is enough to showe that $\forall \varepsilon > 0$ $T D_E(0,1)$ can be covered by a finite union of balls of radius ε.

Fix $\varepsilon > 0$ and let $S \in K(E,F)$ s.t.

$$\| T - S \|_{\mathcal{L}(E,F)} < \frac{\varepsilon}{2} .$$

Since S is compact, $\exists$ a finite cover

$$S D_E(0,1) \subseteq \bigcup_{j=1}^{N} D_F\left(f_j, \frac{\varepsilon}{2}\right)$$

$$\Rightarrow \quad T D_E(0,1) \subseteq \bigcup_{j=1}^{N} D_F(f_j, \varepsilon)$$

Indeed $\forall x \in D_E(0,1)$ $\exists f_j$ s.t. $\| Sx - f_j \|_F < \frac{\varepsilon}{2}$

$$\|Tx - f_j\|_F \le \|Tx - Sx + Sx - f_j\|_F \le$$

$$\le \|Tx - Sx\|_F + \|Sx - f_j\|_F$$

$$\le \underbrace{\|T - S\|_{\mathcal{L}(E,F)}}_{< \frac{\varepsilon}{2}} \overbrace{\|x\|_E}^{\le 1} + \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow T D_E(0,1)$ is sequentially relatively compact $\Rightarrow \overline{T D_E(0,1)}$ is compact

Theorem X, Y B spaces.

Then $T \in \mathcal{K}(X, Y) \iff T^* \in \mathcal{K}(Y', X')$

Pf $\Longrightarrow$

Assume $T \in \mathcal{K}(X, Y)$, we need to show $\overline{T^* D_{Y'}(0,1)}$ is compact in X'

$K = \overline{T D_X(0,1)}$ in Y K is compact in Y.

Let $\{y'_n\}$ be any sequence in $D_{Y'}(0,1)$

$\{y'_n|_K\}$ is a sequence in $C^0(K, \mathbb{R})$.

By Ascoli-Arzela there exists a subsequence and $\varphi \in C^0(K, \mathbb{R})$ s.t.

$$y'_n|_K \longrightarrow \varphi \quad \text{in } C^0(K, \mathbb{R})$$

$$\lim_{n \to +\infty} \sup_{x \in D_X(0,1)} \left| \langle y'_n, Tx \rangle_{Y' \times Y} - \varphi(Tx) \right| = 0$$

$$\leq \lim_{n \to +\infty} \sup_{y \in K} \left| \langle y'_n, y \rangle_{Y' \times Y} - \varphi(y) \right| = 0$$

$$\lim_{n \to +\infty} \sup_{x \in D_X(0,1)} \left| \langle T^* y'_n, x \rangle_{X' \times X} - \varphi(Tx) \right| = 0$$

$y'_n \in D_{Y'}(0,1)$

$$\| T^* y'_n - T^* y'_m \|_{X'} \xrightarrow{n,m \to +\infty} 0$$

$$\sup_{x \in D_X(0,1)} \left| \langle T^* y'_n - T^* y'_m, x \rangle_{X' \times X} \right| \xrightarrow{n,m \to +\infty} 0$$

$$\| T^* y'_n - T^* y'_m \|_{X'} \xrightarrow{n,m \to +\infty} 0$$

$\{y'_n\}$ in $D_{Y'}(0,1)$

Ty'_n is convergent in X'

We showed $T \in K(\cancel{X}, Y)$

$\Rightarrow T^* \in K(Y', X')$

Let $T \in \mathcal{L}(X, Y)$, $T^* \in K(Y', X')$

$\Rightarrow T^{**} \in K(X'', Y'')$

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, hook, "J_X"'] & Y \arrow[d, hook, "J_Y"] \\
X'' \arrow[r, "T^{**}"'] & Y''
\end{tikzcd}$$

$T^{**} J_X = J_Y T$ ✓

$$\langle T^{**} J_X x, y' \rangle_{Y'' \times Y'} = \langle J_X x, T^* y' \rangle_{X'' \times X'}$$

$$= \langle x, T^* y' \rangle_{X \times X'} = \langle Tx, y' \rangle_{Y \times Y'}$$

$$= \langle J_Y T x, y' \rangle_{Y'' \times Y'} \quad \forall y' \in Y'.$$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook, "J_X"'] & Y \arrow[d, hook, "J_Y"] \\ X'' \arrow[r, "T^{**}"'] & Y'' \end{tikzcd}$$

$$T^{**} J_X = J_Y T \quad \checkmark$$

$$T^{**}|_{R(J_X)} \in K(R(J_X), Y'')$$

$$T^{**}|_{R(J_X)} \in K(R(J_X), R(J_Y))$$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "J_X \, S"'] & Y \arrow[d, shift left, "S"] \\ R(J_X) \arrow[r, "T^{**}|_{R(J_X)}"'] & R(J_Y) \arrow[u, shift left, "J_Y^{-1}"] \end{tikzcd}$$

$$T = J_Y^{-1} \circ T^{**}|_{R(J_X)} \circ J_X$$

compact $\Longrightarrow$ T compact

Theorem Let F be a Hilbert space and E be B-space.
Then $T \in \mathcal{L}(E, F)$ is compact if and only if $T = \lim_{n \to +\infty} T_n$ in $\mathcal{L}(E, F)$ with T_n of finite rank.

Pf We only need to prove that $T \in K(E,F)$
$\Rightarrow \exists \{T_n\}$ as in the statement

Let $\varepsilon > 0$, we know

$$T D_E(0,1) \subseteq \bigcup_{j=1}^{N} D_F(f_j, \varepsilon)$$

let $G = \text{span}\{f_1, \dots, f_N\}$
$\dim G < +\infty$

$\forall x \in D_E(0,1)$ $\exists j$ s.t.
$\|Tx - f_j\|_F < \varepsilon$

Let P_G be the orthogonal projection in F on G.

$$\|P_G Tx - f_j\|_F \leq \|Tx - f_j\|_F < \varepsilon$$

$$\|P_G Tx - Tx\|_F \leq \|P_G Tx - f_j + f_j - Tx\|_F$$

$$\leq \|P_G Tx - f_j\|_F + \|f_j - Tx\|_F < 2\varepsilon$$

$$\forall x \in D_E(0,1)$$

$$\Rightarrow \|P_G T - T\|_{\mathcal{L}(E,F)} \leq 2\varepsilon$$

Since P_G is of finite rank $P_G T$ is finite rank. $\varepsilon = \frac{1}{n}$