Prop Let $K \subseteq H$ K closed and convex.

$\exists \quad P_K : H \to K$ (·

$f \in H \longrightarrow u = P_K f \in K$

$\|u - f\|_{\cancel{H}} \leq \|v - f\| \quad \forall v \in K$

$\|P_K f - P_K g\| \leq \|f - g\|$ $(P_K \circ P_K = P_K)$

Corollary If K is closed vector space then $P_K \in \mathcal{L}(H)$ and is a projection, $P_K^2 = P_K$.

Theorem $\forall f \in H' \; \exists y \in H$ s.t.

$$\langle f, x \rangle_{H' \times H} = (x, y)_H .$$

$$(\cdot, y)_H$$

Pf $T : H \to H'$

$$y \longmapsto (\cdot, y)_H$$

$\|x\| \le 1$

$$\left| \langle T y, x \rangle_{H' \times H} \right| = \left| (x, y)_H \right| \le \|y\|_H \|x\|_H$$

$$\le \|y\|_H$$

$$\|Ty\|_{H'} \le \|y\|_H \qquad \|T\| \le 1$$

$$\|Ty\|_{H'} \|y\|_H \ge \left| \langle Ty, y \rangle_{H' \times H} \right| = \left| (y, y)_H \right| = \|y\|_H^2$$

$$\|Ty\|_{H'} \ge \|y\|_H \qquad \|Ty\|_{H'} = \|y\|_H$$

T is an isometry $\qquad R(T) = TH \subseteq H'$

Since T is an isometry $H \longrightarrow TH$

$\Rightarrow$ TH is complete and so is closed inside H'.

If by contradiction $TH \subsetneq H'$ we know

$\exists \; h \in H''$, $h \neq 0$ s.t. $\langle Ty, h \rangle_{H' \times H''} = 0 \quad \forall y \in H.$

H is reflexive $\Rightarrow$ $J : H \hookrightarrow H''$ is in fact an isomorphism so $\exists \, x \in H$ ($x \neq 0$) s.t. $h = Jx$

$$0 = \langle Ty, Jx \rangle_{H' \times H''} = \langle Ty, x \rangle_{H' \times H} = 0$$

$\forall y \in H$

$$= (x, y)_H = 0 \quad \forall y \in H$$

$$y = x \quad \Rightarrow \quad (x, x)_H = \|x\|^2 = 0 \Rightarrow x = 0$$

$TH = H'$ so any $f \in H'$ is of the form $f(x) = (x, y)_H$ for a $y \in H$

Def A subset $S \subset H$ is called orthonormal if

$\forall\, x \in S \quad \|x\|_H = 1$ and

if $x, y \in S \quad x \neq y$ then $(x, y)_H = 0$

Ex 1 $L^2([-\pi,\pi]^d) \qquad n \in \mathbb{Z}^d \qquad x \in [-\pi,\pi]^d$

$$\left\{ \frac{e^{in\cdot x}}{(2\pi)^{\frac{d}{2}}} \right\}_{n \in \mathbb{Z}^d} \text{ is an orthonormal set}$$

and it is an orthonormal basis of $L^2([-\pi,\pi]^d)$

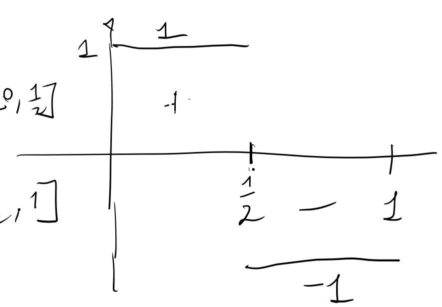
Ex 1 Rademacher function

In $L^2([0,1])$

$\{ r_k \}_{k=0,1,2\ldots\ldots}$

$r_0 = 1$

$$r_1 = \begin{cases} 1 & \text{in } [0,\frac{1}{2}] \\ -1 & [\frac{1}{2},1] \end{cases}$$



$$r_k(x) = (-1)^{j-1} \quad \text{if } x \in [2^{-k}(j-1), 2^{-k}j] \qquad [0,1] = \bigcup_{j=1}^{2^k} [2^{-k}(j-1), 2^{-k}j]$$

$$0 = \langle r_0, r_1 \rangle = \int_0^1 r_0(x)\, r_1(x)\, dx = \int_0^1 r_1(x)\, dx = 0$$

$0 \le k < j$

$\langle r_k, r_j \rangle = 0$

$\text{span}\, \{ r_k \}_{k \in \mathbb{N} \cup \{0\}} \subset L^p([0,1]) \qquad \forall\, p < \infty$

$2 \le p < +\infty$

Theorem Let $S \subset H$ be orthonormal.

1) $\forall\ u \in H$

$$\sum_{s \in S} |(u, s)_H|^2 \leq \|u\|_H^2 \qquad \text{(Bessel inequality)}$$

2) Let $V_S = \overline{\mathrm{Span}(S)}$. The following are equiv.

a) $u \in V_S$

b) $\sum_{s \in S} |(u, s)_H|^2 = \|u\|_H^2$

c) $\sum_{s \in S} (u, s)_H\, s$ ~~is~~ this is a convergent series in H with limit $= u$

3) $\forall\ u \in H$ $\sum_{s \in S} (u, s)_H\, s$ is a convergent series in V_S with limit $P_{V_S} u$

$$\sum_{s \in S} |(u, s)_H|^2 = \|P_{V_S} u\|_H \qquad \text{(Parseval identity)}$$

$u \in H$ $\qquad$ $L^2(\mathbb{R}^d)$

Pf let $\operatorname{card} S \leq \operatorname{card} \mathbb{N}$.

$S = \{s_j\}_{j \in \mathbb{N}}$

$s_1, \dots, s_n$

$$S_n u = \sum_{j=1}^{n} (u, s_j)_H \, s_j$$

$$\|S_n u\|_H^2 = (S_n u, S_n u)_H = \sum_{j,k=1}^{n} (u, s_j)(u, s_k)\underbrace{(s_j, s_k)_H}_{\delta_{jk}}$$

$$= \sum_{j=1}^{n} |(u, s_j)|^2$$

$$\|u - S_n u\|_H^2 = \|u\|_H^2 - \|S_n u\|_H^2$$

$$(u - S_n u, u - S_n u) = \|u\|^2 - 2(S_n u, u) + \|S_n u\|^2$$

$-2\|S_n u\|^2$

$$(S_n u, u) = \|S_n u\|^2$$

$$\left(\sum_{j=1}^{n} (u, s_j) s_j, u\right) = \sum_{j=1}^{n} (u, s_j)(s_j, u) =$$

$$= \sum_{j=1}^{n} |(u, s_j)|^2 = \|S_n u\|_H^2$$

$$\|S_n u\|^2 = \|u\|^2 - \|u - S_n\|^2 \leq \|u\|^2$$

$$\|S_n u\|^2 \leq \|u\|^2$$

$$\sum_{j=1}^{n} |(u, s_j)|^2 \leq \|u\|^2$$

$$\sum_{s \in S} |(u, s)|^2 \leq \|u\|^2 \qquad \operatorname{card} S \leq \operatorname{card} \mathbb{N}$$

$\operatorname{card} S > \operatorname{card} \mathbb{N}$ $\qquad \forall u$

$$\operatorname{card} \{s \in S : (u, s) \neq 0\} \leq \operatorname{card} \mathbb{N}.$$

If not $\exists$ $m \in \mathbb{N}$ s.t.

$$\operatorname{card} \{s \in S : |(u, s)| \geq \tfrac{1}{m}\} = +\infty$$

If $\forall$ $m \in \mathbb{N}$

$$\operatorname{card} \{s \in S : |(u, s)| \geq \tfrac{1}{m}\} < +\infty$$

$$\Rightarrow \{s \in S : (u, s) \neq 0\} = \bigcup_{m=1}^{\infty} \{s \in S : |(u, s)| \geq \tfrac{1}{m}\}$$

a) $\Rightarrow$ c)

$u \in V_S \Rightarrow \sum_{s \in S} (u,s)\, s = u$

Let $u \in V_S$ $\forall\, \varepsilon_n \xrightarrow{} 0$ $\exists$ $s_{\sigma_1}, \dots, s_{\sigma_k} \in S$

and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t.

$$\| u - \sum_{j=1}^{k} \lambda_j s_{\sigma_j} \| \le \varepsilon_n.$$

$S' \subseteq S$

$u \in V_{S'}$ $\quad S' = \{ s_j \}_{j \in \mathbb{N}}$

$$S_n u = \sum_{j=1}^{n} (u, s_j)\, s_j$$

$$\| u - \sum_{j=1}^{n} \lambda_j s_j \|^2 = \| \underbrace{u - \sum_{j=1}^{n} (u,s_j) s_j}_{u - S_n u} + \sum_{j=1}^{n} \big((u,s_j) - \lambda_j\big) s_j \|^2$$

$$= \| u - S_n u \|^2 + \sum_{j=1}^{n} |(u,s_j) - \lambda_j|^2$$

$$\Big(u - S_n u + \sum_{j=1}^{n} \big((u,s_j)-\lambda_j\big) s_j \Big) = \| u - S_n u\|^2 + \| \sum_{j=1}^{n} \big((u,s_j)-\lambda_j\big) s_j \|^2$$

$$+ 2 \Big(u - S_n u, \sum_{j=1}^{n} \big((u,s_j)-\lambda_j\big) s_j \Big) = 0$$

$$\Big(= \sum_{j=1}^{n} \big((u,s_j)-\lambda_j\big) \underbrace{\big(u - S_n u, s_j\big)}_{0} \Big) \qquad 1 \le j \le n$$

$$(u - S_n u, s_j) = (u, s_j) - \sum_{k=1}^{n} \big((u,s_k) s_k, s_j \big)$$

$$= (u, s_j) - \sum_{k=1}^{n} (u, s_k)\, \delta_{kj} = 0$$

$$\left\| u-\sum_{j=1}^{n} \lambda_j s_j \right\|^2 = \left\| u-\sum_{j=1}^{n}(u,s_j)s_j + \sum_{j=1}^{n}\left((u,s_j)-\lambda_j\right)s_j \right\|^2$$

$$= \|u - S_n u\|^2 + \underbrace{\sum_{j=1}^{n} |(u,s_j)-\lambda_j|^2}_{\geq 0}$$

$$\geq \|u - S_n u\|^2$$

$$\|u - S_n u\| \leq \left\| u-\sum_{j=1}^{n}\lambda_j s_j \right\| \leq \varepsilon$$

this holds for any n s.t. $\{s_1, \dots, s_n\} \supseteq \{s_{\sigma 1}, \dots, s_{\sigma n}\}$

$$\lim_{n\to+\infty} S_n u = u$$

we proved $\sum_{s\in S} (u,s)_H \, s \overset{(\varepsilon)}{=} u \Rightarrow u \in V_S := \overline{\operatorname{span}\{S\}}$

$\Downarrow$

b) $\sum_{s\in S} |(u,s)|^2 = \|u\|^2$

$$\sum_{j=1}^{\infty} |(u,s_j)|^2 = \|u\|^2$$

$$\|u\|^2 = \|S_n u\|^2 + \underbrace{\|u - S_n u\|^2}_{n\to+\infty}$$

$$\boxed{\|u\|^2 = \lim_{n\to+\infty} \|S_n u\|^2} = \lim_{n\to+\infty} \sum_{j=1}^{n} |(u,s_j)|^2 = \sum_{j=1}^{\infty} |(u,s_j)|^2$$

(b) $\Leftrightarrow$ $\|u\|^2 = \lim_{n\to+\infty} \|S_n u\|^2$

$$\|u - S_n u\|^2 = \|u\|^2 - \|S_n u\|^2 \xrightarrow{n\to+\infty} 0$$

$\Rightarrow \lim_{n\to+\infty} S_n u = u$ in H

Def Given a H an orthonormal basis is an orthonormal se S s.t. $V_S = H$.

Thm Every H has an orthonormal basis.

Pf $\mathfrak{S} = \{ S : S \text{ is orthonormal in } H \}$

in $\mathfrak{S}$ the inclusion of set defines a partial order. The conclusion from Zorn's lemma is about the existence of maximal elements in $\mathfrak{S}$.

To get the conclusion we need to know that given any chain in $\mathfrak{S}$ there exists a majorant of the chain.

Let $\mathcal{C} = \{ S_j \}_{j \in J}$

$S = \bigcup_{j \in J} S_j$. S is orthonormal

$\forall s \in S$, $\exists j \in J$ s.t. $s \in S_j$

$\|s\| = 1$

Let $s, \sigma \in S$ $s \neq \sigma$ $\exists j \in J$ s.t. $s \in S_j$

$\exists k \in J$ s.t. $\sigma \in S_k$

It not reductive to assume that $s, \sigma \in S_k$

$(s, \sigma) = 0$

Let S be maximal in $\mathfrak{S}$

$V_S \subsetneq H$ $\Rightarrow$ it is easy to see that

$\exists y \neq 0$ in H $(\, , y)$ s.t. $(x, y) = 0$ $\forall x \in V_S$

$\|y\| = 1$ $\Rightarrow$ $(s, y) = 0$ $\forall s \in S$

and $S \cup \{y\}$ is a strictly larger than S, a contradiction.

$$\left\{ \frac{e^{in\cdot x}}{(2\pi)^{\frac{d}{2}}} \right\}_{n\in\mathbb{Z}^n}$$ is an ~~ort~~ orthonormal basis

in $L^2([-\pi,\pi]^d) \ni f$

$$f = \sum_{n\in\mathbb{Z}^d} \frac{\left(f, e^{in\cdot x}\right)}{(2\pi)^{\frac{d}{2}}} \frac{e^{inx}}{(2\pi)^{\frac{d}{2}}}$$

$$(f,g) = \int_{[-\pi,\pi]^d} f(x)\,\overline{g(x)}\,dx$$

$$\frac{1}{(2\pi)^d}\left(f, e^{inx}\right) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]} f(x)\, e^{-inx}\, dx = \hat{f}(n)$$

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbb{Z}^d} \frac{|\hat{f}(n)|^2}{(2\pi)^d}$$

$$f \in L^2(\mathbb{T}^d, \mathbb{C}) \longrightarrow \hat{f} \in \ell^2(\mathbb{Z}^d, \mathbb{C})$$

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \|\hat{f}\|_{\ell^2}$$