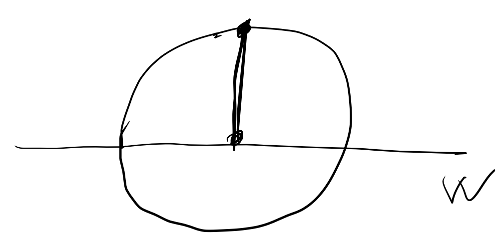
12 december

<u>Theorem</u> If X is a B-space and $\overline{D_X(0,1)}$ is compact, then $\dim X < +\infty$

<u>Lemma</u> Let X be a B space and $W \subsetneq X$ closed. Then $\exists \{v_n\}$ in X $\|v_n\| = 1$ s.t. $\text{dist}(v_n, W) \xrightarrow{n \to +\infty} 1$

<u>Pf</u>



<u>Corollary</u> Let $\dim X = +\infty$ and consider a strictly increasing sequence $\{E_n\}$ of closed vector spaces in X. Then $\exists \{x_n\}$ in X with $\|x_n\| = 1$ s.t. $x_n \in E_n$ $\forall n$ and $\text{dist}(x_n, E_{n-1}) \geq \frac{1}{2}$ $\forall n$.

Theorem (Fredholm alternative)

Let X be B-space and $K \in \mathcal{K}(X)$ and $T = 1 - K$. Then

1) $\dim \ker T < +\infty$

2) $R(T) = (\ker T^*)^{\perp}$

3) $\ker T = 0 \iff R(T) = X$

4) $\dim \ker T = \dim \ker T^*$

Remark 2) is important because it says that $(1-K)x = x_0$ has a solution iff $x_0 \in (\ker T^*)^{\perp}$

$$X = X_0 \bigoplus_{\lambda \in \sigma(K) \setminus \{0\}} N_g(K - \lambda)$$

$T = 1 - K$

$$\sigma(K|_{X_0}) = \{0\}$$

$(1-K)x = f \qquad f \in R(T)$

$(1-K)x = f_0 \qquad$ in X_0

$x = (1-K)^{-1} f_0 \qquad$ in X_0

$(1-K)x = f_\lambda \qquad N_g(K-\lambda)$

$$\sigma((1-K)|_{N_g(K-\lambda)}) = \{\lambda\} \quad \text{if } \lambda \neq 1$$

$x = (1-K)^{-1} f_\lambda \qquad$ in $N_g(K-\lambda)$

Let us consider $\lambda = 1$

$\lambda = 1$ in $N_g(K - 1)$ which has finite dimension the operator $K - 1$ is like a sum of Jordan blocks

Let us consider a Jordan block of dimension 3

$$T = K - 1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{sp}\{e_1, e_2, e_3\}$$

In X' we can consider $\mathrm{sp}\{e'_1, e'_2, e'_3\}$

s.t. $\boxed{\langle e_j, e'_k \rangle_{X \times X'} = \delta_{jk}}$

$$X' = \mathrm{sp}^{\perp}\{e_1, e_2, e_3\} \oplus \mathrm{sp}\{e'_1, e'_2, e'_3\}$$

$$T^* = K^* - 1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R(T) = \ker T^*$$

$R(T) = \mathrm{sp}\{e_1, e_2\}$

$\langle e_1, e'_3 \rangle = \langle e_2, e'_3 \rangle = 0 \qquad \ker T^* = \mathrm{sp}\{e'_3\}$

Theorem Let $K \in \mathcal{K}(X)$ $\dim X = +\infty$ Then

1) $0 \in \sigma(K)$ $\Big($ If $0 \notin \sigma(K) \Rightarrow K^{-1} \in \mathcal{L}(X)$ and K would be an isomorphism

2) $\lambda \in \sigma(K)$ with $\lambda \neq 0 \Rightarrow$ λ is an eigenvalue of K.

3) Either $\sigma(K)$ is finite or $\sigma(K) \setminus \{0\}$ can be written as a sequence convergent to 0.

4) Each $\lambda \in \sigma(K) \setminus \{0\}$ has finite algebraic multiplicity.

$\Big($ $N_g(K-\lambda) = \bigcup_{n=1}^{\infty} \ker\big((K-\lambda)^n\big)$ is an increasing sequence, $\dim N_g(K-\lambda)$ is the algebraic dimension of λ $\Big)$

$(K-\lambda)^n x = 0 \Rightarrow (K-\lambda)^{n+1} x = 0$

$(K-\lambda)(K-\lambda)^n x$

$K\overline{D_X(0,1)}$ would be closed and compact (because K is compact)

$\exists\, c > 0$ st $D_X(0,c) \subseteq K\overline{D_X(0,1)}$

$\overline{\text{would}}$ be compact

$\overline{D_X(0,1)}$ compact $\Rightarrow \dim X < +\infty$

If $\lambda \in \sigma(K) \setminus \{0\}$ if λ is not an eigenvalue

$\ker(K-\lambda) = 0 \Rightarrow R(K-\lambda) = X$

$K - \lambda = \lambda\left(\frac{K}{\lambda} - 1\right) = -\lambda\left(1 - \frac{K}{\lambda}\right)$

~~We can use~~ $K-\lambda$ is injective, continuous and surjective $\Rightarrow$ $K-\lambda$ is an isomorphism in X

$\Rightarrow (K-\lambda)$ has a bounded inverse $\Rightarrow \lambda \in \rho(K)$

But this contradicts the hypothesis $\lambda \in \sigma(K)$

suppose that and $\{\lambda_n\}$ is a sequence in $\sigma(K)$

s.t. $\lambda_n \xrightarrow{\neq 0} \lambda \neq 0$

$$K x_n = \lambda_n x_n$$

$$X_n = \text{span}\{x_1, \dots, x_n\}$$

$$y_n \in X_n \setminus X_{n-1} \qquad \text{dist}(y_n, X_{n-1}) > \frac{1}{2}$$

$$\|y_n\| = 1$$

$n > m$

$$\left\| \frac{K y_n}{\lambda_n} - \frac{K y_m}{\lambda_m} \right\| = \left\| y_n + \left[\underset{\substack{\uparrow \\ X_m}}{-y_m} + \underset{\substack{\uparrow \\ X_{n-1}}}{\frac{(K-\lambda_n) y_n}{\lambda_n}} - \underset{\substack{\uparrow \\ X_m}}{\frac{(K-\lambda_m) y_m}{\lambda_m}} \right] \right\|$$

$$\geq \text{dist}(y_n, X_{n-1}) > \frac{1}{2}$$

$$K \frac{y_n}{\lambda_n}$$

we,

Examples

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad L^p(0,1)$$

It is not a bounded operator for $p=1$

$$\chi_n = n\,1_{[0,\frac{1}{n}]} \qquad |\chi_n|_{L^1} = 1$$

$$T\chi_n = n\,\frac{1}{x}\int_0^x 1_{[0,\frac{1}{n}]}(t)\,dt =$$

$$= \begin{cases} n & \text{if } x \le \frac{1}{n} \\ \frac{1}{x} & \text{if } x \ge \frac{1}{n} \end{cases}$$

$$\int_0^1 T\chi_n = \int_0^{\frac{1}{n}} T\chi_n + \int_{\frac{1}{n}}^1 T\chi_n =$$

$$= 1 + \lg(n) \xrightarrow{n\to+\infty} +\infty$$

$1 < p < +\infty$

$t = sx \qquad dt = x\,ds$

$$\Big\| x^{-1} \underbrace{\int_0^x f(t)\,dt}\Big\|_{L^p(0,1)} =$$

$$= \Big\| \int_0^1 f(sx)\,ds \Big\|_{L^p(0,1)}$$

$$\leq \int_0^1 \| f(s\cancel{x}) \|_{L^p_x(0,1)}\,ds$$

$$\left(\frac{1}{s} \int_0^1 |f(sx)|^p\,dxs \right)^{\frac{1}{p}} \qquad y = xs$$

$$= \left(\frac{1}{s} \int_0^s |f(y)|^p\,dy \right)^{\frac{1}{p}} \leq \left(\frac{1}{s} \int_0^1 |f(y)|^p\,dy \right)^{\frac{1}{p}}$$

$$\| f(s\,\cdot) \|_{L^p_x(0,1)} \leq s^{-\frac{1}{p}} \| f \|_{L^p(0,1)}$$

$$\leq \underbrace{\int_0^1 s^{-\frac{1}{p}}\,ds}_{\frac{1}{1-\frac{1}{p}}} \| f \|_{L^p(0,1)} \qquad 1 < p < +\infty$$

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad 1 < p < +\infty$$

$$\delta_\lambda f(x) = \lambda^{\frac{1}{p}} f(\lambda x)$$

$$T\delta_\lambda f(x) = \frac{1}{x}\int_0^x \lambda^{\frac{1}{p}} f(\lambda t)\,dt =$$

$s = \lambda t$

$ds = \lambda\, dt$

$dt = \frac{1}{\lambda} ds$

$$= \lambda^{\frac{1}{p}} \frac{1}{\lambda x}\int_0^{\lambda x} f(s)\,ds$$

$$= \delta_\lambda (Tf)(x) \qquad T\delta_\lambda = \delta_\lambda T$$

$$\delta_\lambda f \xrightarrow{\lambda \to 0^+} 0 \quad \text{in } \sigma(L^p, L^{p'})$$

If T is compact

$$T\delta_\lambda f \xrightarrow{\lambda \to 0^+} 0$$

$$\delta_\lambda T f \nearrow$$

$$\overbrace{\|\delta_\lambda T f\|_{L^p(I)}}^{\neq 0} = \|Tf\|_{L^p} \iff$$

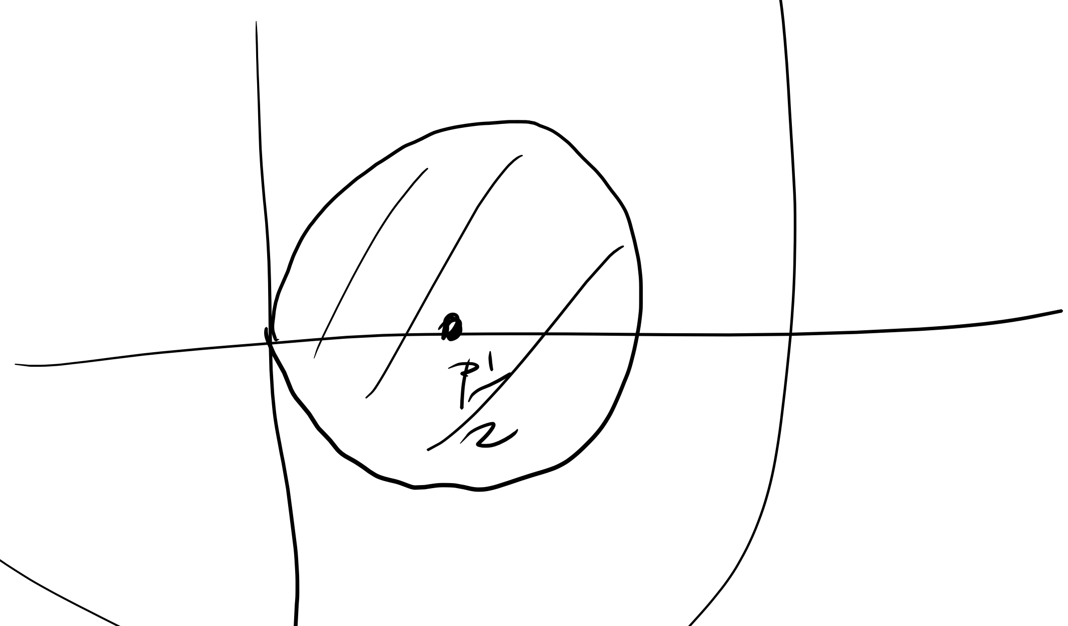
$Tf = 0$ false if $f > 0$

$1 < p < +\infty$

The set of eigenvalues of T is

$D_{\mathbb{C}}\left(\frac{p'}{2}, \frac{p'}{2}\right)$

$$\sigma(T) = \overline{D_{\mathbb{C}}\left(\frac{p'}{2}, \frac{p'}{2}\right)}$$



$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

If $f \in \ker T$ $f \not\equiv 0$

$$Tf = 0 \Rightarrow \int_0^x f(t)\,dt = F(x) \equiv 0 \quad f \in L^p(0,1) \subseteq L^1(0,1)$$

a.e in $(0,1)$

$$F'(x) = f(x) = 0 \Rightarrow f \equiv 0 \quad \text{a}$$

contradiction $\Rightarrow$ 0 is not an eigenvalue

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt = z\,f(x)$$

$$\int_0^x f(t)\,dt = z\,x\,f(x)$$

$$f(x) = z\,x\,f'(x) + z\,f(x)$$

$$z\,x\,f' + (z-1)f = 0$$

$$f' + \frac{z-1}{zx} f = 0$$

$$\mu\left(\cancel{f'} + \frac{z-1}{\cancel{z}x}\right)f = (\mu f)' =$$

$$= \cancel{\mu f'} + \mu' f$$

$$\mu' = \frac{z-1}{zx}\mu \qquad \mu = e^{\int \frac{z-1}{zx}dx}$$

$$\mu(x) = e^{\frac{z-1}{z}\lg x} = x^{\frac{z-1}{z}}$$

$$\left(x^{\frac{z-1}{z}} f\right)' = 0 \qquad x^{\frac{z-1}{z}} f(x) = 1$$

$$f(x) = x^{-\frac{z-1}{z}}$$

$$|f(x)|^p = \left|x^{-\frac{z-1}{z}}\right|^p = x^{-\mathrm{Re}\left(\frac{z-1}{z}\right)p}$$

$$\mathrm{Re}\left(\frac{z-1}{z}\right)p < 1$$

$$\mathrm{Re}\left(1 - \frac{1}{z}\right) < \frac{1}{p} \qquad \frac{1}{p'} = 1 - \frac{1}{p}$$

$$z = z_R + i z_I$$

$$\frac{1}{p'} < \mathrm{Re}\left(\frac{1}{z}\right) =$$

$$= \mathrm{Re}\,\frac{\bar z}{|z|^2} = \frac{z_R}{z_R^2 + z_I^2}$$

$$\frac{1}{p'} < \frac{z_R}{z_R^2 + z_I^2}$$

$$z_R^2 + z_I^2 - p' z_R < 0$$

$$z_R^2 + z_I^2 - 2p'\frac{z_R}{2} + \left(\frac{p'}{2}\right)^2 - \left(\frac{p'}{2}\right)^2 < 0$$

$$\left(z_R - \frac{p'}{2}\right)^2 + z_I^2 < \left(\frac{p'}{2}\right)^2 \qquad D_{\mathbb{C}}\left(\frac{p'}{2}, \frac{p'}{2}\right)$$

$$\sigma(T) \underset{\supseteq}{=} \overline{D_{\mathbb{C}}\left(\frac{p'}{2}, \frac{p'}{2}\right)}$$

If $z \notin \overline{D_{\mathbb{C}}\left(\frac{p'}{2}, \frac{p'}{2}\right)} \Rightarrow z \in \rho(T)$

$(T-z)^{-1}$ is bounded operator

$$(T-z)f = g \qquad g \in C_c^\infty(0,1)$$

$$f(x) = -\frac{1}{z}g(x) - x^{\frac{1}{z}-1}\,\frac{1}{z}\left(1 - \frac{1}{z}\right)\int_0^x t^{-\frac{1}{z}} g(t)\,dt$$

$$=: S_z g$$

S_z extends into a bounded operator in $L^p(0,1)$

$(T-z)S_z g = g$ for all $g \in L^p(0,1)$