Lezione 43 - 15/12/2025

T. (fondamentale)

$f \in \mathcal{R}([a,b])$, $x_0 \in [a,b]$, f sia continua in x_0

detta $F(x) = \int_a^x f(t)\,dt$ funzione integrale

Allora F è derivabile in x_0 e $F'(x_0) = f(x_0)$

Corollario $f : [a,b] \to \mathbb{R}$, f continua su $[a,b]$

f continua $\Rightarrow f \in \mathcal{R}$

Allora F è derivabile su $[a,b]$, e $F' = f$.

Teorema (Torricelli)

se f è continua su $[a,b]$ e G è una sua primitiva

Allora $\int_a^b f(t)\,dt = G(b) - G(a)$

G primitiva di f significa G derivabile e $G' = f$

Conseguenza

per calcolare $\int_a^b f(t)\,dt$ si cerca una primitiva di f !

Cercare primitive

1) leggere a rovescio la tabella delle derivate (facendo attenzione ai domini)

	f	una primitiva	
	a	ax	$(\mathbb{R})$
$n \in \mathbb{N}$,	x^n	$\frac{1}{n+1} x^{n+1}$	$(\mathbb{R})$
$m \in \mathbb{Z}, m<0, m \neq -1$,	x^m	$\frac{1}{m+1} x^{m+1}$	$(]-\infty,0[$ o $]0,+\infty[)$
$\alpha \in \mathbb{R}\setminus\{0\}$	x^α	$\frac{1}{\alpha+1} x^{\alpha+1}$	$(]0,+\infty[)$
	$\frac{1}{x}$	$\log x$	$(]0,+\infty[)$
	$\frac{1}{x}$	$\log(-x)$	$(]-\infty,0[)$
	e^x	e^x	$(\mathbb{R})$
	$\operatorname{sen} x$	$-\cos x$	
	$\cos x$	$\operatorname{sen} x$	
	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcsen} x$	$(]-1,1[)$
	$\frac{1}{1+x^2}$	$\operatorname{arctg} x$	$(\mathbb{R})$

se ne conosco una le conosco tutte (basta sommare una costante)

$\frac{1}{x}$	$\log\lvert x\rvert$

$$\int_0^1 (e^x + x + \sqrt{x})\,dx =$$

$\sqrt{x} = x^{\frac{1}{2}}$

$x^\alpha \qquad \frac{1}{\alpha+1} x^{\alpha+1}$

$$\int (e^x + x + \sqrt{x})\,dx = e^x + \frac{1}{2}x^2 + \frac{1}{\frac{1}{2}+1} x^{3/2} + c = e^x + \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + c$$

$$= \left. e^x + \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} \right]_0^1 = e^1 + \frac{1}{2}\cdot 1 + \frac{2}{3}\cdot 1 - \left(e^0 + \frac{1}{2}\cdot 0 + \frac{2}{3}\cdot 0\right)$$

$$= e + \frac{1}{2} + \frac{2}{3} - 1 = e + \frac{3+4-6}{6} = e + \frac{1}{6}$$

Tecniche di integrazione

1) integrazione per parti

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supponiamo che F, G siano 2 funzioni derivabili con derivate continue su I intervallo,

$$(FG)' = F'G + FG'$$

quindi FG è primitiva di $F'G + FG'$ che è continua

quindi il teor. di Torricelli

$$\int_a^b (F'G + FG')(t)\,dt = (FG)(b) - (FG)(a)$$

$$\int_a^b \left(F'(t)G(t) + F(t)G'(t)\right) dt = F(b)G(b) - F(a)G(a)$$

$$\boxed{\int_a^b F'(t)G(t) = F(b)G(b) - F(a)G(a) - \int_a^b F(t)G'(t)\,dt}$$

a livello di primitive (cioè di integrali indefiniti)

$$\boxed{\int F'(t)G(t)\,dt = F\,G - \int F(t)G'(t)\,dt}$$

come si usa?

$$\int_0^\pi \underset{G}{x}\cdot \underset{F'}{\operatorname{sen} x}\,dx = \underset{G}{x}\cdot\underset{F}{(-\cos x)}\Big|_0^\pi - \int_0^\pi 1\cdot(-\cos x)\,dx$$

$$= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx$$

$$= -x\cos x + \operatorname{sen} x\Big|_0^\pi$$

$$= (-\pi\cos\pi + \operatorname{sen}\pi) - (-0\cdot\cos 0 + \operatorname{sen} 0)$$

$$= (\pi + 0) - (0) = \pi$$

abbiamo ottenuto $-x\cos x + \operatorname{sen} x$ è primitiva di $x \operatorname{sen} x$

conviene fare una verifica!

derivo $-x\cos x + \operatorname{sen} x$

$-\cos x - x(-\operatorname{sen} x) + \cos x$

$-\cancel{\cos x} + x\operatorname{sen} x + \cancel{\cos x}$

$x \operatorname{sen} x$

OK

(avrei potuto

$$\int_0^\pi \underset{F'}{x}\,\underset{G}{\operatorname{sen} x}\,dx = \underset{F}{\frac{x^2}{2}}\cdot\underset{G}{\operatorname{sen} x}\Big|_0^\pi - \int_0^\pi \frac{x^2}{2}(-\cos x)\,dx$$

$$= \frac{x^2}{2}\operatorname{sen} x\Big|_0^\pi + \int_0^\pi \frac{x^2}{2}\cos x\,dx$$

più complicato)

$$\int_1^2 \underset{G}{x}\,\underset{F'}{e^x}\,dx = \underset{G}{x}\cdot\underset{F}{e^x}\Big|_1^2 - \int_1^2 e^x\,dx$$

$$= xe^x\Big|_1^2 - e^x\Big|_1^2 = xe^x - e^x\Big|_1^2$$

$$= (2e^2 - e^2) - (e^1 - e^1) = e^2$$

- $$\int_1^2 \lg x\,dx = \int_1^2 1\cdot \lg x\,dx$$

$$\int_1^2 \lg x\,dx = \int_1^2 \underset{F'}{1}\cdot \underset{G}{\lg x}\,dx = \underset{F}{x}\,\underset{G}{\lg x}\Big|_1^2 - \int_1^2 \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$$

$$= x\lg x\Big|_1^2 - \int_1^2 1\,dx$$

$$= x\lg x - x\Big|_1^2$$

$$= 2\lg 2 - 2 - (1\cancel{\lg 1} - 1)$$

$$= 2\lg 2 - 1$$

$$\boxed{\int \lg x\,dx = x\lg x - x + c}$$

verifica

$$(x\lg x - x)' = \lg x + \cancel{x\frac{1}{x}} \cancel{- 1} = \lg x \quad \text{ok}$$

- $\int \operatorname{sen}^2 x\,dx$

$$\int \underset{F'}{\operatorname{sen} x}\cdot \underset{G}{\operatorname{sen} x}\,dx = \underset{F}{-\cos x}\cdot \underset{G}{\operatorname{sen} x} - \int(-\cos x)\cdot\cos x$$

$$= -\cos x\operatorname{sen} x + \int \cos^2 x\,dx$$

$$\int \operatorname{sen}^2 x\,dx = -\cos x\operatorname{sen} x + \underbrace{\int \cos^2 x\,dx} \qquad \cos^2 x = 1 - \operatorname{sen}^2 x$$

$$\int \cos^2 x\,dx = \int \underset{F'}{\cos x}\cdot \underset{G}{\cos x}\,dx = \underset{F}{\operatorname{sen} x}\cdot \underset{G}{\cos x} - \int \underset{F\ G'}{\operatorname{sen} x(-\operatorname{sen} x)}\,dx$$

$$= \operatorname{sen} x\cos x + \int \operatorname{sen}^2 x\,dx$$

$$\int \operatorname{sen}^2 x\,dx = -\cancel{\cos x\operatorname{sen} x} + \cancel{\operatorname{sen} x\cos x} + \int \operatorname{sen}^2 x\,dx$$

$$\int \operatorname{sen}^2 x\,dx = -\cos x\operatorname{sen} x + \int 1 - \operatorname{sen}^2 x\,dx$$

$$= -\cos x\operatorname{sen} x + \int 1\cdot dx - \int \operatorname{sen}^2 x\,dx$$

$$2\int \operatorname{sen}^2 x\,dx = -\cos x\operatorname{sen} x + x + \text{cost}$$

$$\boxed{\int \operatorname{sen}^2 x\,dx = \frac{x - \cos x\operatorname{sen} x}{2} + \text{cost}}$$

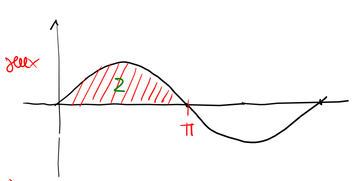
verifica

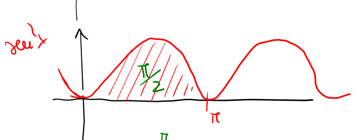
$$\left(x - \frac{\cos x\operatorname{sen} x}{2}\right)' = \frac{1}{2}\left(1 - (-\operatorname{sen}^2 x + \cos^2 x)\right)$$

$$= \frac{1}{2}(1 + \operatorname{sen}^2 x - \cos^2 x)$$

$$= \frac{1}{2}\left(\operatorname{sen}^2 x + \underbrace{(1 - \cos^2 x)}_{=\operatorname{sen}^2 x}\right)$$

$$= \frac{1}{2}(2\operatorname{sen}^2 x) = \operatorname{sen}^2 x$$





$$\int_0^\pi \operatorname{sen} x\,dx = -\cos x\Big|_0^\pi = 2$$

$$= -\cos(\pi) - (-\cos 0)$$

$$= 1 + 1$$

$$\int_0^\pi \operatorname{sen}^2 x\,dx = \frac{x - \cos x\operatorname{sen} x}{2}\Big|_0^\pi$$

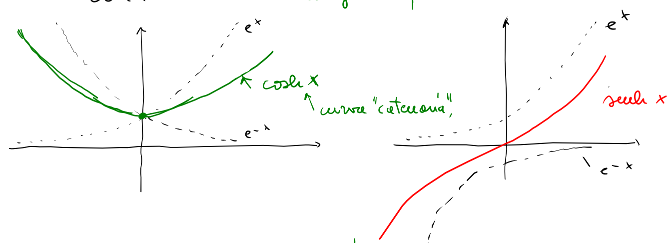
$$\frac{1}{2}(\pi - 0) - \frac{1}{2}(0 - 0) = \frac{\pi}{2}$$

Funzioni iperboliche

def.
$$\mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{e^x + e^{-x}}{2} = \cosh x$$
coseno iperbolico di x

$$\mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{e^x - e^{-x}}{2} = \operatorname{senh} x$$
seno iperbolico di x

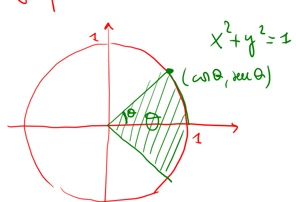
$$\frac{\operatorname{senh} x}{\cosh x} = \operatorname{tgh} x$$
tangente iperbolica



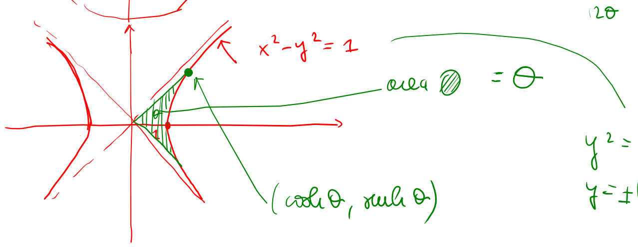
$$(\operatorname{senh})'(x) = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$(\cosh)'(x) = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \operatorname{senh} x$$

$$(\operatorname{senh})'' = \operatorname{senh} \qquad \left(\operatorname{sen}'' = -\operatorname{sen}\right)$$
$$(\cosh)'' = \cosh \qquad \left(\cos'' = -\cos\right)$$

¿ perché si chiamano così ?



area = θ
(ampio θ)



$$y^2 = x^2 - 1$$
$$y = \pm\sqrt{x^2-1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

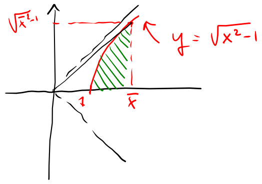
/ f. di Taylor con resto di Lagrange per far vedere che la serie converge.

$$e^{i\theta} \overset{def}{=} 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!}$$
$$= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots\dots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \dots\right)}_{\operatorname{sen}\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i \operatorname{sen}\theta}$$ formula di <u>Eulero</u>

$z \in \mathbb{C}$
$$\boxed{e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4} + \dots}$$

Esercizio



$$\int_1^{\bar x}\sqrt{t^2-1}\,dt$$

$$\left((x^2-1)^{\frac12}\right)'$$
$$\frac{1}{\cancel{2}}\cdot(x^2-1)^{-\frac12}\,\cancel{2}x$$
$$\frac{x}{\sqrt{x^2-1}}$$

considero $f(x)=\lg\left(x-\sqrt{x^2-1}\right)$

calcolo $f'(x) = -\dfrac{1}{\sqrt{x^2-1}}$

$$f'(x)=\frac{1}{x-\sqrt{x^2-1}}\cdot\left(1-\frac{x}{\sqrt{x^2-1}}\right)=\frac{1}{\cancel{x-\sqrt{x^2-1}}}\cdot\frac{\cancel{\sqrt{x^2-1}-x}^{\,-1}}{\sqrt{x^2-1}}=$$

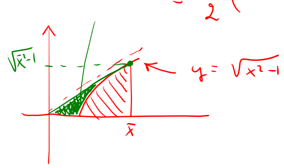
f	primitiva
$-\frac{1}{\sqrt{x^2-1}}$	$\lg(x-\sqrt{x^2-1})$

$$\int\sqrt{x^2-1}\,dx=\int \underset{F'}{1}\cdot\underset{G}{\sqrt{x^2-1}}=x\sqrt{x^2-1}-\int x\cdot\frac{x}{\sqrt{x^2-1}}\,dx$$
$$=x\sqrt{x^2-1}-\int\frac{x^2}{\sqrt{x^2-1}}\,dx$$
$$=x\sqrt{x^2-1}-\underbrace{\int\frac{x^2-1}{\sqrt{x^2-1}}\,dx}_{\sqrt{x^2-1}}-\int\frac{1}{\sqrt{x^2-1}}\,dx$$

$$2\int\sqrt{x^2-1}\,dx=x\sqrt{x^2-1}-\int\frac{1}{\sqrt{x^2-1}}\,dx$$

$$\boxed{\int\sqrt{x^2-1}\,dx=\frac12\left(x\sqrt{x^2-1}+\lg\left(x-\sqrt{x^2-1}\right)\right)+c}$$

$$\int_1^{\bar x}\sqrt{x^2-1}\,dx=\frac12\left(x\sqrt{x^2-1}+\lg\left(x-\sqrt{x^2-1}\right)\right)\Bigg|_1^{\bar x}$$
$$=\frac12\left(\bar x\sqrt{\bar x^2-1}+\lg\left(\bar x-\sqrt{\bar x^2-1}\right)\right)$$



area (verde) = triangolo − area (rossa)

$$=\frac{\bar x\cdot\sqrt{\bar x^2-1}}{2}-\frac12\bar x\sqrt{\bar x^2-1}-\frac12\lg\left(\bar x-\sqrt{\bar x^2-1}\right)$$

$$\boxed{\text{area (verde)}=-\frac12\lg\left(x-\sqrt{x^2-1}\right)}$$
$$\| $$
$$\frac{\theta}{2}$$

$$\theta=-\lg\left(x-\sqrt{x^2-1}\right)$$

quanto vale x? risultato $\boxed{x=\dfrac{e^{\theta}+e^{-\theta}}{2}}$