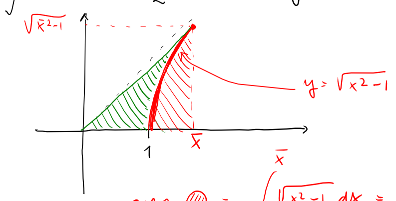
Lezione 44 - 16/12/2025

$$\int \sqrt{x^2-1}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} + \log\left(x-\sqrt{x^2-1}\right)\right) + c$$

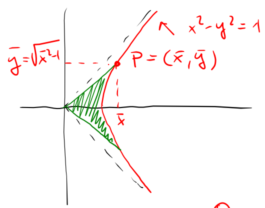


area ▨ $= \displaystyle\int_1^{\bar{x}} \sqrt{x^2-1}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1}+\log\left(x-\sqrt{x^2-1}\right)\right)\Big|_1^{\bar{x}}$

$= \frac{1}{2}\left(\bar{x}\sqrt{\bar{x}^2-1}+\log\left(\bar{x}-\sqrt{\bar{x}^2-1}\right)\right)$

area ▨ + area ▨ $= \dfrac{\bar{x}\cdot\sqrt{\bar{x}^2-1}}{2}$

area ▨ $= \dfrac{-\log\left(\bar{x}-\sqrt{\bar{x}^2-1}\right)}{2}$ $\quad\left(0<\bar{x}-\sqrt{\bar{x}^2-1}<1\right)$ verificare!



area ▨ $=\theta$

trovare le coordinate di P in funzione di θ

cioè quanto vale $\bar{x}$ in funzione di θ?

$$\theta = -\log\left(\bar{x}-\sqrt{\bar{x}^2-1}\right)$$

$$-\theta = \log\left(\bar{x}-\sqrt{\bar{x}^2-1}\right)$$

$$e^{-\theta} = \bar{x}-\sqrt{\bar{x}^2-1}$$

$$\sqrt{\bar{x}^2-1} = -e^{-\theta}+\bar{x}$$

$$\cancel{\bar{x}^2}-1 = e^{-2\theta}-2e^{-\theta}\bar{x}+\cancel{\bar{x}^2}$$

$$2e^{-\theta}\bar{x} = e^{-2\theta}+1$$

$$\boxed{\bar{x} = \frac{e^{\theta}+e^{-\theta}}{2}} \qquad \bar{x}=\cosh\theta$$

$$\bar{y}=\sqrt{\bar{x}^2-1} = \sqrt{\frac{e^{2\theta}+2+e^{-2\theta}}{4}-1}$$

$$= \sqrt{\frac{e^{2\theta}-2+e^{-2\theta}}{4}} = \sqrt{\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2}$$

$$\bar{y} = \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta$$

~ ● ~

Integrazione per sostituzione

sia $g:[\alpha,\beta]\longrightarrow[a,b]$ con g di classe C^1 (continua, derivabile con derivata continua)

$g(\alpha)=a,\ g(\beta)=b$

$f:[a,b]\longrightarrow\mathbb{R}$, con f continua

poniamo $F(x)=\displaystyle\int_a^x f(t)\,dt$ $\qquad$ F è la funzione integrale

considero $\varphi(t)=F(g(t))$ $\qquad$ $\varphi:[\alpha,\beta]\longrightarrow\mathbb{R}$

considero $\underline{\varphi(t) = F(g(t))}$ $\qquad \varphi : [\alpha, \beta] \to \mathbb{R}$

$$\varphi(t) = \int_{a = g(\alpha)}^{g(t)} f(y)\,dy = \int_{g(\alpha)}^{g(t)} f(y)\,dy \qquad \varphi(\alpha) = \int_{g(\alpha)}^{g(\alpha)} f(y)\,dy = 0 \qquad \varphi(\beta) = \int_{g(\alpha)}^{g(\beta)} f(y)\,dy$$

chi è $\varphi'(t)$?

$$\varphi'(t) = F'(g(t)) \cdot g'(t)$$

chi è F' ? f

quindi $\varphi'(t) = \underbrace{\underbrace{f}_{\text{cont.}}(\underbrace{g(t)}_{\text{cont.}})}_{\text{continua}} \underbrace{g'(t)}_{\text{continua}}$ è continua

allora $\displaystyle\int_\alpha^\beta \varphi'(t)\,dt = \varphi(\alpha) - \varphi(\beta) = \int_{g(\alpha)}^{g(\beta)} f(t)\,dt$

$$\int_\alpha^\beta f(g(t))\,g'(t)\,dt =$$

concludo

$$\boxed{\int_a^b f(t)\,dt = \int_\alpha^\beta f(g(s))\,g'(s)\,ds \qquad \text{dove } a = g(\alpha),\ b = g(\beta)}$$

formula di integrazione per sostituzione

se per caso g è invertibile $\qquad \alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$

$$\boxed{\int_a^b f(t)\,dt = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(s))\,g'(s)\,ds}$$

come si usa? $\qquad$ devo calcolare $\displaystyle\int_a^b f(t)\,dt$

"cambio la variabile" → devo mettere la g "giusta"

ponendo $t = g(s)$

al posto di dt devo mettere $g'(s)ds$

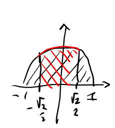
scrivo "$dt = g'(s)ds$"

se t vale a, $\quad g(s) = a$, $s = g^{-1}(a)$

t vale b $\quad g(s) = b$, $s = g^{-1}(b)$

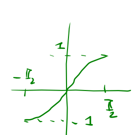
$$\int_a^b f(t)\,dt \quad \text{diventa} \quad \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(s))\,g'(s)\,ds$$

ES. $\displaystyle\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1 - x^2}\,dx \quad \left(= \frac{\pi}{2}\right)$ $\qquad y = \sqrt{1 - x^2}$

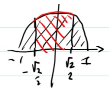


cambio la variabile $\qquad x = \operatorname{sen}\theta$

$$[-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1]$$
$$\theta \longmapsto \operatorname{sen}\theta$$

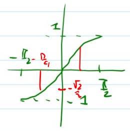


ES. $= \displaystyle\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx \quad \left(= \frac{\pi}{2}\right)$ $\qquad y = \sqrt{1-x^2}$



cambio la variabile $\quad x = \operatorname{sen}\theta$

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$
$$\theta \longmapsto \operatorname{sen}\theta$$



$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1-(\operatorname{sen}\theta)^2}\cdot\cos\theta\,d\theta \qquad dx = (\operatorname{sen}\theta)'\,d\theta = \cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta\cdot\cos\theta\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$$\int \cos^2\theta\,d\theta = \frac{\theta + \operatorname{sen}\theta\cos\theta}{2} + c$$

$$= \left.\frac{\theta + \operatorname{sen}\theta\cos\theta}{2}\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\frac{\pi}{4} + \operatorname{sen}\frac{\pi}{4}\cos\frac{\pi}{4}}{2} - \left(\frac{-\frac{\pi}{4} + \operatorname{sen}\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{4}\right)}{2}\right)$$

$$= \frac{\frac{\pi}{4} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{2} - \left(\frac{-\frac{\pi}{4} + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}$$

$$\left((1-x^2)^{\frac{1}{2}}\right)' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

In altra maniera

$$\int \sqrt{1-x^2}\,dx = \int \underset{F'}{1}\cdot\underset{G}{\sqrt{1-x^2}} = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

per parti

$$= x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} + \int \frac{x^2-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \operatorname{arcsen}x + c$$

concludere $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \operatorname{arcsen}x\right) + c$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx = \left.\frac{1}{2}\left(x\sqrt{1-x^2} + \operatorname{arcsen}x\right)\right|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2} + \operatorname{arcsen}\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\sqrt{1-\left(-\frac{\sqrt{2}}{2}\right)^2} + \operatorname{arcsen}\left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\right) - \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2} - \frac{\pi}{4}\right)$$

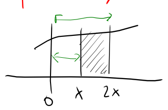
$$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$= \frac{1}{2} + \frac{\pi}{4}$$

ES. $\lim_{x\to 0} \dfrac{\int_x^{2x} e^{-t^2}\,dt}{x}$ $\qquad \dfrac{0}{0}$

chiamo $F(x)=\int_0^x e^{-t^2}\,dt$ ← F ?

posso esprimere $\int_x^{2x} e^{-t^2}\,dt$ usando F ?



$\int_x^{2x} = \int_0^{2x} - \int_0^x = F(2x)-F(x)$

$\lim_{x\to 0} \dfrac{F(2x)-F(x)}{x}$ con $F(x)=\int_0^x e^{-t^2}\,dt$ $\qquad \dfrac{0}{0}$

$\overset{H}{=} \lim_{x\to 0} \dfrac{2F'(2x)-F'(x)}{1}$ $\qquad (F(2x))' = F'(2x)\cdot 2$

$F'(x)=e^{-x^2}$

$= \lim_{x\to 0} 2e^{-(2x)^2} - e^{-x^2} = 2-1 = 1$

$\lim_{x\to 1} \dfrac{\int_x^{2x-1} \frac{1}{1+\lg t}\,dt}{x-1}$ $\qquad \int_1^x \dfrac{1}{1+\lg t}\,dt = F(x)$

$\int_x^{2x-1} \dfrac{1}{1+\lg t}\,dt = F(2x-1)-F(x)$ $\qquad F'(x)=\dfrac{1}{1+\lg x}$

$\lim_{x\to 1} \dfrac{F(2x-1)-F(x)}{x-1}$ $\qquad \dfrac{F(1)-F(1)}{0}=\dfrac{0}{0}$

$\overset{H}{=} \lim_{x\to 1} \dfrac{F'(2x-1)\cdot 2 - F'(x)}{1} =$

$\lim_{x\to 1} 2\underbrace{\dfrac{1}{1+\lg(2x-1)}}_{\to 1} - \underbrace{\dfrac{1}{1+\lg x}}_{\to 1} = 2-1 = 1$

$\lim_{x\to 0} \dfrac{\int_{x^3}^{x^2} (1-t^2)e^{-t^2}\,dt}{x\cdot \sin x}$ $\qquad F(x)=\int_0^x (1-t^2)e^{-t^2}\,dt$

$= \lim_{x\to 0} \dfrac{F(x^2)-F(x^3)}{x\cdot \underbrace{\sin x}_{\to 1}}\cdot\dfrac{x}{x}$

$(F(x^2))' = F'(x^2)\cdot 2x$

$(F(x^3))' = F'(x^3)\cdot 3x^2$

resta da calcolare $\lim_{x\to 0} \dfrac{F(x^2)-F(x^3)}{x^2}$

$\overset{H}{=} \lim_{x\to 0} \dfrac{2x\,F'(x^2) - 3x^2 F'(x^3)}{2x}$

$\lim_{x\to 0} F'(x^2) - \dfrac{3}{2}x\,F'(x^3)$

$F'(t) = (1-t^2)e^{-t^2}$ $\qquad F'(x^2) = (1-x^4)e^{-x^4}$

$F'(x^3) = (1-x^6)e^{-x^6}$

$$\lim_{x\to 0} F'(x^2) - \frac{3}{2}x F'(x^3)$$

$F'(t) = (1-t^2)e^{-t^2}$ $\quad F'(x^2) = (1-x^4)e^{-x^4}$

$F'(x^3) = (1-x^6)e^{-x^6}$

$$\lim_{x\to 0} \underbrace{(1-x^4)e^{-x^4}}_{1} - \frac{3}{2}\cdot\underbrace{x}_{0}\cdot \underbrace{(1-x^6)e^{-x^6}}_{1} = 1$$

ES.

$$\int_1^2 x \lg x \, dx = \frac{x^2}{2}\lg x\Big|_1^2 - \int_1^2 \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx$$

(x: F', $\lg x$: G)

$$= \frac{x^2}{2}\lg x\Big|_1^2 - \int_1^2 \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\lg x - \frac{x^2}{4}\Big|_1^2$$

verifica

$\int x\lg x\,dx = \frac{x^2}{2}\lg x - \frac{x^2}{4} + c$

$\left(\frac{x^2}{2}\lg x - \frac{x^2}{4}\right)' = x\cdot\lg x + \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}} - \frac{x}{2} = x\lg x + \cancel{\frac{x}{2}} - \cancel{\frac{x}{2}} = x\lg x$

OK

$$= 2\lg 2 - 1 - \left(0 - \frac{1}{4}\right) = 2\lg 2 - \frac{3}{4}$$

$$\int_0^1 \frac{e^x}{e^x+1}\,dx$$

$e^x = t$

$e^x dx = dt$

$x=0 \quad t=1$

$x=1 \quad t=e$

$(\lg(e^x+1))' = \frac{e^x}{e^x+1}$

$$= \int_1^e \frac{dt}{t+1}$$

$\int \frac{1}{t+1}\,dt = \lg(1+t)$

$$= \lg(1+t)\Big|_1^e = \lg(e+1) - \lg 2$$

$$\int \frac{e^x}{e^x+1}\,dx$$

$e^x = t$

$e^x dx = dt$

$$= \int \frac{dt}{t+1} = \lg(1+t) + c = \lg(1+e^x) + c$$

$$\int x \cos^2 x\,dx$$

(x: F, $\cos^2 x$: G')

$\int \cos^2 x\,dx = \frac{x + \cos x\cdot \sin x}{2} + c$

$\left(\frac{\sin^2 x}{4}\right)' = \frac{2\sin x\cos x}{2}$

$$= x\cdot\left(\frac{x+\cos x\sin x}{2}\right) - \int \frac{x+\cos x\sin x}{2}\,dx$$

$$= \frac{x^2}{2} + \frac{x\cos x\sin x}{2} - \int\frac{x}{2}\,dx - \int\frac{\cos x\sin x}{2}\,dx$$

$$= \frac{x^2}{2} + \frac{x\cos x\sin x}{2} - \frac{x^2}{4} - \frac{\sin^2 x}{4} + c$$

$$= \frac{x^2}{4} + \frac{x\cos x\sin x}{2} - \frac{\sin^2 x}{4} + c$$

$$\int \log(1+x^2)\,dx$$

$$= x\log(1+x^2) - \int x \cdot \frac{1}{1+x^2} \cdot 2x \quad dx$$

$$= x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2\int \cancelto{1}{\frac{x^2+1}{x^2+1}} - \frac{1}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2x + 2\int \frac{1}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2x + 2\,\mathrm{arctg}\,x + c.$$