18 december

Definition H a hilbert space on $\mathbb{R}$ (or $\mathbb{C}$)
A bilinear map $B: H \times H \to \mathbb{R}$ is bounded if $\exists$ $\gamma > 0$ st.

$$|B(x,y)| \leq \gamma \|x\| \|y\|,$$

and it is coercive if $\exists$ $\delta > 0$
st. $|B(x,x)| \geq \delta \|x\|^2$.

$s \in \mathbb{R}$ $\mathbb{T}^d$ I will define
$H^s(\mathbb{T}^d)$

$f: \mathbb{T}^d \to \mathbb{C}$, $\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx$
$n \in \mathbb{Z}^d$

we now that
$f \in L^2(\mathbb{T}^d, \mathbb{C}) \to \hat{f} \in \ell^2(\mathbb{Z}^d, \mathbb{C})$ is an isomorphism.

$H^s(\mathbb{T}^d, \mathbb{C})$ is the completion of the space of trigonometric polynomials
($f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{inx}$ where only finitely many $\hat{f}(n) \neq 0$) with norm

$$\|f\|^2_{H^s} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2$$
$\langle n \rangle = \sqrt{1 + |n|^2}$ (Japanese bracket)

Sobolev spaces.

$$\widehat{\partial_x^\alpha f}(n) = (in)^\alpha \hat{f}(n)$$

$$f \to \partial_x^\alpha f \implies \hat{f} \to (in)^\alpha \hat{f}(n)$$

$(in)^\alpha = (in_1)^{\alpha_1} \cdots (in_d)^{\alpha_d}$ $n = (n_1, \dots, n_d)$, $\alpha = (\alpha_1, \dots, \alpha_d)$

$$H^1(\mathbb{T}^d) = \{ f \in L^2(\mathbb{T}^d) : \nabla f \in L^2(\mathbb{T}^d) \}$$

$\widehat{\nabla f}(n) = in \hat{f}(n) : \mathbb{Z}^d \to \mathbb{C}^d$

$\nabla f \in L^2(\mathbb{T}^d) \iff in\hat{f} \in \ell^2(\mathbb{Z}^d)$

It is easy to show that the previously defined norm $\|f\|_{H^1(\mathbb{T}^d)}$ is a norm
equivalent to $\sqrt{\|f\|^2_{L^2(\mathbb{T}^d)} + \|\nabla f\|^2_{L^2(\mathbb{T}^d)}}$

$\|f\|_{L^2(\mathbb{T}^d)} + \|\nabla f\|_{L^2(\mathbb{T}^d)}$

H^1 is a hilbert space
$$(f,g)_{H^1} = (\nabla f, \nabla g)_{L^2} + (f,g)_{L^2}$$
$$= \sum_{j=1}^d (\partial_j f, \partial_j g)_{L^2} + (f,g)_{L^2}$$

Let now $\{a_{ij}(x)\}_{i,j=1}^{d}$ be a matrix with

$a_{ij} \in L^\infty(\mathbb{T}^d, \mathbb{R}) \qquad a_{ij} = a_{ji}$

$$c|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j \leq C\,|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^d$$

$0 < c < C < +\infty$

$$B(u,v) = \sum_{i,j=1}^{d} (a_{ij}\,\partial_j u, \partial_i v) + \langle V u, v\rangle$$

$V \in L^\infty(\mathbb{T}^d, \mathbb{R}) \qquad V \geq 1$ a.e.

Then $$|B(u,u)| \leq \left| \int_{\mathbb{T}^d} \sum_{i,j} a_{ij}(x)\,\partial_j u\,\partial_i u \,dx\right| + \left|\int_{\mathbb{T}^d} V u u\, dx\right|$$

$$\leq C \int_{\mathbb{T}^d} |\nabla u|^2\,dx + |V|_{L^\infty} \int_{\mathbb{T}^d} |u|^2 dx$$

$$\leq \left(C + |V|_{L^\infty}\right)\left(\underbrace{\int_{\mathbb{T}^d} (|\nabla u|^2 + |u|^2)\,dx}_{\|u\|^2_{H^1}}\right)$$

$$|B(u,u)| \leq \underbrace{(C + |V|_{L^\infty})}_{K}\ \|u\|^2_{H^1}$$

It is easy to see that B is symmetric

$$\Rightarrow |B(u,v)| \leq K\ \|u\|_{H^1}\,\|v\|_{H^1}$$

$$B(u,u) = \int_{\mathbb{T}^d} \underbrace{\sum_{i,j=1}^{d} a_{ij}\,\partial_i u\,\partial_j u}_{\geq c\,|\nabla u|^2}\,dx$$

$$+ \int_{\mathbb{T}^d} V\, u^2\,dx \qquad V \geq 1$$

$$\geq \int_{\mathbb{T}^d} \left(c\,|\nabla u|^2\,dx + |u|^2\right)dx$$

$$\geq \min\{c, 1\}\ \|u\|^2_{H^1}$$

So our $$B(u,v) = \sum_{i,j=1}^{d} \left\{(a_{ij}\,\partial_j u, \partial_i v) + \right.$$

$$+ \langle V u, v\rangle$$

Theorem Let B be bounded coercive. Then $\exists\, S \in \mathcal{L}(H)$ s.t. $S^{-1} \in \mathcal{L}(H)$ s.t.

$$B(x, Sy) = (x, y) \qquad \forall\, x, y \in H.$$

$$\Big(B(x,y) = (x, S^{-1} y) \Big)$$

We have $\|S\|_{\mathcal{L}(H)} \le \delta^{-1}$ and $\|S^{-1}\|_{\mathcal{L}(H)} \le M$.

If furthermore B is symmetric then S is symmetric.

Pf $D := \{ y \in H : \exists\, y^* \in H$ s.t. $(x,y) = B(x, y^*) \ \forall x \in H \}$

We will prove $D = H$.

$D \neq \emptyset$ indeed $0 \in D$ $\quad 0^* = 0$

Let $y \in D$. Then y^* is unique. If there are two distinct y_1^* and y_2^*

$\forall x \in H \quad (x, y) = B(x, y_j^*) \qquad j = 1, 2$

$$\Rightarrow B(x, y_2^* - y_1^*) = 0 \qquad \forall x \in H$$

also for $x = y_2^* - y_1^*$

$$\delta \|y_2^* - y_1^*\|^2 \le B(y_2^* - y_1^*, y_2^* - y_1^*) = 0$$
$$\Rightarrow y_2^* = y_1^*$$

Since y^* is uniquely defined, we have a map

$$S : D \to H \qquad y \to y^*.$$

It is easy to see that it is a linear map.

It is a bounded operator $u \in D$ $u \neq 0$

$$\delta \|Su\|^2 \le |B(Su, Su)| = |(Su, u)| \le$$
$$\le \|Su\| \ \|u\|$$

$$\delta \|Su\| \le \|u\|$$
$$\|Su\| \le \delta^{-1} \|u\|$$

We claim D is closed. $D \subseteq \overline{D}$

$S : D \longrightarrow H$ $\quad$ S continuous

$S : \overline{D} \longrightarrow H$

Let $z \in \overline{D}$. $\exists\, \underset{D}{\underset{\cap}{y_n}} \xrightarrow{n \to +\infty} z$

$$(x, y_n) = B(x, Sy_n) \qquad \forall n$$

$$\Big\downarrow n \to +\infty \qquad\qquad \Big\downarrow n \to +\infty$$

$$(x, z) = B(x, Sz) \qquad \forall x \in H$$

$\Rightarrow z \in D$ and $Sz = z^*$

We need to prove now $D = H$.

If not $D \subsetneq H$ $\quad \exists\, w_0 \in D^\perp$

$w_0 \neq 0$.

$x \longrightarrow B(x, w_0)$ is a functional in H

and $\exists\, w \in H$ s.t. $B(x, w_0) = (x, w)$ $\forall x \in H$,

so that $w_0 = Sw$ and so $w \in D$.

$$\delta \|w_0\|^2 \le |B(w_0, w_0)| = |(w_0, w)| = 0$$

$\Rightarrow w_0 = 0$.

Furthermore $\|S\|_{\mathcal{L}(H)} \le \delta^{-1}$

$$B(x,y) = (x, S^{-1}y)$$

$$(S^{-1}y, S^{-1}y) = B(S^{-1}y, y)$$

$$\|S^{-1}y\|^2 \leq |B(S^{-1}y, y)| \leq \gamma \, \|S^{-1}y\| \, \|y\|$$

$$\|S^{-1}y\| \leq \gamma \, \|y\| \Rightarrow \|S^{-1}\| \leq \gamma.$$

B symmetric $\Rightarrow$ S symmetric (complete the proof)

$H^s(\mathbb{T}^d)$

$$(f,g)_{H^s} = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \hat{f}(n)\overline{\hat{g}(n)}$$

Abstractly these spaces are all isomorphic

$$\begin{tikzcd}
L^2(\mathbb{T}^d) \arrow[r] \arrow[ddd, bend right, leftrightarrow] & H^s(\mathbb{T}^d) \arrow[ddd, bend left, leftrightarrow] \\
f \arrow[r] \arrow[d] & {} \\
\hat{f} \arrow[r] & \langle n\rangle^{-s}\hat{f} \arrow[u] \\
\ell^2(\mathbb{Z}^d) \arrow[r] & \ell^{2,s}(\mathbb{Z}^d)
\end{tikzcd}$$

were

$$\ell^{2,s}(\mathbb{Z}^d) := \{ g : \mathbb{Z}^d \to \mathbb{C} : \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |g(n)|^2 < +\infty \}$$

$$\left(H^s(\mathbb{T}^d)\right)' = ?$$

Here it is natural to identify $\left(H^s(\mathbb{T}^d)\right)'$ with $H^{-s}(\mathbb{T}^d)$

by means of $(\ ,\ )_{L^2(\mathbb{T})}$

Indeed for any pair $f \in H^s(\mathbb{T}^d)$ and $g \in H^{-s}(\mathbb{T}^d)$ $\quad f \to \hat{f}$

are in $\ell^2(\mathbb{Z}^d)$

$$(f,g)_{L^2(\mathbb{T})} = (\hat{f},\hat{g})_{\ell^2(\mathbb{Z}^d)} = (\langle n\rangle^{s}\hat{f}, \langle n\rangle^{-s}\hat{g})_{\ell^2(\mathbb{Z}^d)}$$

$$f \in H^s(\mathbb{T}^d),\ g \in H^{-s}(\mathbb{T}^d) \longrightarrow (f,g)_{L^2(\mathbb{T}^d)}$$

$$H^s(\mathbb{T}^d) \times H^{-s}(\mathbb{T}^d) \longrightarrow \mathbb{C}$$

$$H^{-s}(\mathbb{T}^d) \hookrightarrow \left(H^s(\mathbb{T}^d)\right)'$$

Theorem (Lax–Milgram) For B like above let $f' \in H'$ and consider the problem of finding $u \in H$ s.t.

$$B(v,u) = \langle v, f' \rangle_{H \times H'} \quad \forall v \in H \quad (*)$$

Then $\exists!\ u \in H$ solving $*$ and is given by $u = Sf$ where $f \in H$ is s.t. $\langle\ , f' \rangle_{H \times H'} = (\ , f)_H$.

Example $H^1(\mathbb{T}^d)$ $H^{-1}(\mathbb{T}^d)$

$$B(v,u) = \sum_{i,j=1}^{d} (a_{ij} \partial_j v, \partial_i u)_{L^2} + (V v, u)_{L^2}$$

$$= (v, f)_{L^2} \qquad \forall v \in H^1(\mathbb{T}^d)$$

$$f \in H^{-1}(\mathbb{T}^d)$$

$$\Big(v, -\sum_{i,j=1}^{d} \partial_j a_{ji} \partial_i u + V u\Big)_{L^2(\mathbb{T}^d)} = (v, f) \qquad \forall v \in L^2(\mathbb{T}^d)$$

$$-\sum_{i,j=1}^{d} \partial_j a_{ji} \partial_i u + V u = f$$

Pf $f' \in H'$ $\exists! f \in H$ s.t.

$$\langle v, f' \rangle_{H \times H'} = (v, f).$$

For $u = Sf$ we know

$$(v, f) = B(v, u) \quad \forall v \in H$$

Therefore we get

$$B(v, u) = \langle v, f' \rangle_{H \times H'} \quad \forall v \in H$$

The following is a classical theorem

Theorem Let H be a separable Hilbert space over $\mathbb{R}$ and $A \in \mathcal{L}(H)$ symmetric ($(Ax,y)=(x,Ay)$ $\forall x,y \in H$) and compact

Then $\exists$ a sequence of real numbers $\{\mu_n\}$ with $\lim_{n\to+\infty} \mu_n = 0$ and an orthonormal basis of H $\{e_n\}_{n\in\mathbb{N}}$ with $Ae_n = \mu_n e_n$.

Theorem Consider

$$B(v,u) = \sum_{i,j} (\partial_j v, a_{ji} \partial_i u)_{L^2(\mathbb{T}^d)} + (\gamma v, u)_{L^2(\mathbb{T}^d)}$$

and ~~consider~~ $= (v,f)_{L^2(\mathbb{T}^d)}$

$$f = Au = \sum_{i,j=1}^{d} -\partial_j a_{ji} \partial_i u + \gamma u \in H^{-1}(\mathbb{T}^d)$$

Then $\exists$ a sequence $\{e_n\}$ in $H^1(\mathbb{T}^d)$ which is a Hilbert basis in $L^2(\mathbb{T}^d)$

s.t. $Ae_n = \lambda_n e_n$ where $\{\lambda_n\}$ is a sequence with $\lim_{n\to+\infty} \lambda_n = +\infty$.

Pf By the previous theorem $\forall f \in L^2(\mathbb{T}^d)$ $\exists$ Sf

$$B(v, Sf) = (v,f)_{L^2(\mathbb{T}^d)} \quad \forall v \in H^1$$

$u \in H^1(\mathbb{T}^d)$

S ~~can be~~ can be thought as a compact operator inside $L^2(\mathbb{T}^d)$