

25 settembre Definition of topological vector space. Balanced and absorbing sets. Proof that there is a basis of balanced neighborhoods. Continuous linear operators between topological vector spaces. Characterization in terms of hyperplanes of linear functionals.

26 settembre Aula 5 C Metrizable topological vector spaces (only statement of theorem). Norms, Normed spaces and Banach spaces. Proof that $C^s(O)$ for O open subset of R^d that $0 < s < 1$ if appropriately defined is a Banach space. Proof that $W^{s,p}(O)$ with $1 \leq p < \infty$ is Banach space.

2 ottobre 2 C H3 Definition of locally convex topological vector spaces X . Proof of the existence of a continuous seminorm, the Minkowski functional, for any open convex neighborhood of 0 in X . Sub-basis of seminorms. Beginning of a discussion on $L^p(0,1)$ when $0 < p < 1$: description of the metric.

3 ottobre Aula 5 C H2bis Verification that the only convex open subset of $L^p(0,1)$ when $0 < p < 1$ is the whole space and that $(L^p(0,1))' = 0$. The case of metrizable locally convex topological vector spaces. Another example: the space of Schwartz test functions. Bounded operators between normed spaces. Notions of strong convergence and of uniform convergence of sequences of operators.

9 Ottobre 2 C H3 Some exercises : proof that the space of Schwartz functions $S(R^d, C)$ with the usual topology does not have a norm; proof that a locally convex space with a countable sub-basis of seminorms is metrizable. Spectrum $\sigma(T)$ and resolvent set $\rho(T)$ of a bounded operator T in a Banach space on C . Analysis of the spectrum of a multiplier operator with m in $BC^0(O)$ with O open subset of R^d . Invertibility of $1+T$ if $\|T\| < 1$ (Neumann series). Proof that $z \in \rho(T)$ if $|z| > \|T\|$.

10 October Aula 5 C H2bis Proof that $R_T(z)$ depends analytically in z . Proof that the spectrum $\sigma(T)$ is non empty. Analysis of the spectrum of a multiplier operator with m in $BC^0(O)$ with O open subset of R^d . An exercise showing that any compact set K in the complex plane coincides with the spectrum of a multiplier operator in a space $L^p(N)$. The exponential of a bounded operator and derivation of the Duhamel formula.

16 Ottobre 2 C H3 Two exercises, one showing that the norm of a multiplier operator with multiplier m in $L^\infty(O)$ has operator norm exactly equal to the $L^\infty(O)$ norm of m , and the second completing the proof that for m in $BC^0(O)$ the spectrum coincides exactly with the closure of $m(O)$ in C . Analytic form of Hahn Banach Theorem (with proof). Proof that in a normed space X for any nonzero x in X there exists a functional f of norm 1 in X such that $f(x) = \|x\|$.

17 October Aula 5 C H2bis Adjoint T' of a bounded linear operator T and proof that for its norm we have $\|T'\| = \|T\|$. Proof of a lemma and of the 1st geometric form of the Hahn-Banach Theorem. Statement (only) of the 2nd geometric version of the Hahn Banach theorem and proof of a corollary of the 2nd geometric version of the Hahn Banach theorem, stating that if Y a closed proper subspace of X then there exists a nonzero functional in X which is zero in Y .

24 ottobre Proof (for normed spaces) of the 2nd geometric version of the Hahn Banach theorem. A concrete application: the Muntz-Szasz Theorem, with a sketch of part of the proof. Statement of the Krein Milman Theorem. Bidual of a normed space and proof that the bidual map is an isometry. Statements of a lemmas about orthogonal and biorthogonal sets.

31 ottobre Definition of Fourier series, in dimension d . Dirichlet Kernel. Proof that there exist continuous functions whose Fourier series are not convergent in 0. The open map theorem. Proof of a corollary that a bijective continuous operator between two Banach spaces has bounded inverse.

Giovedì 6 novembre 2025 Application of the Open Mapping theorem to the fact that $c_0(\mathbb{Z})$ has elements that are not the Fourier coefficients of functions in $L^1(\mathbb{T})$. Closed graph theorem. Complementable spaces. Projections. Spectral projections.

Venerdì 7 novembre 2025 More on direct sums and spectral projections. Definition of $\sigma(E, E')$ topology for E a Banach space. Proof that it is a Hausdorff. Closed convex sets in the strong topology are the same as closed convex sets in the $\sigma(E, E')$ topology. If $\dim E = \infty$ then any open subspace in the $\sigma(E, E')$ topology contains a line. Proof that the closure of the boundary of the unitary sphere of center the origin in the $\sigma(E, E')$ topology and if $\dim E = \infty$ is the closed disk of radius 1.

November 13 Proof that if $\dim E = \infty$ then E with the $\sigma(E, E')$ topology is not metrizable. Some sequences converging weakly to 0. The weak $\sigma(E', E)$ topology

November 13 $\sigma(E, E')$, a lemma of convergence of sequences. Proof that $c_0(\mathbb{N})$ is not a closed subspace of $l^\infty(\mathbb{N})$ for the $\sigma(l^\infty(\mathbb{N}), l^1(\mathbb{N}))$ topology. Proof of the theorem by Banach and Alaoglu. An example of bounded sequence in $(l^\infty(\mathbb{N}))'$ which does not have convergent subsequences in the $\sigma((l^\infty(\mathbb{N}))', l^\infty(\mathbb{N}))$ topology. Definition of reflexive Banach space.

November 20 Proof of Goldstine's Theorem. Proof of the Kakutani theorem. Reflexivity of closed subspace (only stated) and equivalence of reflexivity of E and E' (only stated). Compactness in $\sigma(E, E')$ of bounded convex closed subspaces of E if E is reflexive Existence of absolute minima for $(-\infty, +\infty]$ valued convex and lower continuous functionals ϕ such that $\lim_{x \rightarrow \infty} \phi(x) = +\infty$ defined in a closed convex subspace A of a reflexive space E .

November 21. Separable spaces. Proof that if E' is separable, then E is separable. Proof that $L^\infty(-1, 1)$ is not separable. Proof that the unit ball in E' is metrizable for the $\sigma(E', E)$ topology if and only if E is a separable Banach space. Proof that the unit ball in E is metrizable for the $\sigma(E, E')$ topology if E' is a separable Banach space. Proof that any bounded sequence in a reflexive Banach space E has a

convergent subsequence in the $\sigma(E, E')$ topology. Definition of uniformly convex Banach space. Theorem of Milman and Pettis. Spaces L^p .

November 27 Proof that for $2 \leq p < \infty$ the spaces L^p are uniformly convex and for $1 < p < \infty$ are reflexive. Proof of the Riesz representation theorem for $1 \leq p < \infty$.

December 4 End of proof of the Kolmogorov, Riesz, Fréchet compactness theorem. Hilbert spaces. Cauchy-Schwarz inequality. Proof of the existence of a unique projection on a closed convex subspace of a Hilbert space H .

December 5 Orthonormal sets. Proof of the Bessel inequality. Proof of Parseval identity for orthonormal families.

December 11 Compact operators. Some examples of non compact operators related to convolutions and the Hardy-Littlewood-Sobolev inequality. Proof that the space of compact operators is closed for the uniform topology. Proof that an operator with values in Hilbert space is compact if and only if it is the limit in the uniform topology of a sequence of finite rank operators. Proof that an operator is compact if and only if the adjoint is compact.

December 12 The Fredholm alternative for operators of the form $I - K$ with K a compact operator in a Banach space X . Proof that if $\dim X = \infty$ then 0 belongs to the $\sigma(K)$, that the spectrum is at most countable, with accumulation point 0 and any nonzero element in $\sigma(K)$ is an eigenvalue. A detailed analysis of the operator $\frac{1}{x} \int_0^x f(t) dt$ in $L^p(0,1)$ with is unbounded for $p=1$ and is bounded and not compact for $p>1$, in particular identification of the eigenvalues for $1 < p < \infty$

December 18 Sobolev spaces $H^s(T^d)$ for any real s . Bounded and coercive bilinear functionals. The theorem of Lax and Milgram on the operator associated to a bounded and coercive bilinear functional. The theorem of Lax and Milgram on weak solutions of problems associated to bounded and coercive bilinear functional and to fixed functional. A concrete example in the torus.

