5. Riemann problem for genuinely nonlinear or linearly degenerate vector fields

For 1-d systems

$$u_t + f(u)_x = 0,$$

we want to construct an entropy solution to the Riemann problem

$$u(t=0) = \begin{cases} u^- & x < 0, \\ u^+ & x > 0, \end{cases}$$

when $|u^+ - u^-| \ll 1$. As starting point, observe that the solution will be closed to the linear system

$$u_t + Df(u^-)u_x = u_t + Au_x = 0,$$

which using the base

$$A = \sum_i \lambda_i r_i \ell_i, \quad Ar_i = \lambda_i r_i, \ \ell_i r_j = \delta_{ij}, \ |r_i| = 1,$$

is written as

$$u(t,x) = \sum_i \langle \ell_i, u_0(x + \lambda_i t) \rangle r_i.$$

In particular, the Riemann Problem will have n distinct one dimensional waves, with speed $\sim \lambda_i$ and along the direction r_i.

We need the following assumptions.

Definition 5.1. The 1-d system is *strictly hyperbolic for* $u \in \Omega$ if

$$\inf_{i \neq j} |\lambda_i(u) - \lambda_j(v)| > 0.$$

The *characteristic speed* $\lambda_i(u)$ is *genuinely nonlinear* if

$$D\lambda_i r_i > 0$$

and *linearly degenerate* if

$$D\lambda_i r_i \equiv 0.$$

For the g.n.l. case we orient $r_i(u)$ according to

$$D\lambda_i r_i > 0.$$

We build the building blocks of the solutions.

Rarefaction waves: These are Lipschitz function in $t > 0$ converging in L^1 to u_0: they are clearly entropy solutions. Define the *rarefaction curves* $R_i(s,u)$ solving the ODE

$$\frac{d}{ds} R_i(s,u) = r_i(R_i(s,u)), \quad R_i(0,u) = u.$$

These curves are well defined being solutions to an ODE. The speed of these waves is $\lambda_i(R_i(s,u))$: indeed defining

$$u(t, \lambda_i(R_i(s,u^-))t) = R_i(s,u^-), \quad u_t + \lambda_i u_x = 0, \ u_x (D\lambda_i r_i)t = r_i, \tag{5.1}$$

$$\begin{aligned} u_t + f(u)_x &= u_t + \lambda_i u_x + (Df(u) - \lambda_i \mathrm{id})u_x \\ &= (Df(R_i) - \lambda_i \mathrm{id}) \frac{r_i}{(D\lambda_i r_i)t} = 0, \end{aligned}$$

if the characteristic field is g.n.l.. A necessary condition for the definition of $u(t,x)$ for $t > 0$ is that

$$s \mapsto \lambda_i(R_i(s,u^-))t$$

is invertible, which gives that the rarefaction waves generates a solution through (5.1) if

s the characteristic field is g.n.l. and is increasing.

Shock curves: assume again that λ_i is g.n.l. but now s is decreasing. In this case the above contruction is valid for $t < 0$, while for $t > 0$ the characteristics cross. We are thus locking to a solution in the form of a shock:

$$-s(u^+ - u^-) + f(u^+) - f(u^-) = 0.$$

These are n algebraic equations in the unknown $u^+, \sigma \in \mathbb{R}^{n+1}$: taking the derivative w.r.t. σ, u at $u^+ = u^-$

$$\det \begin{bmatrix} 0 & Df(u^-) - \sigma \mathrm{id} \end{bmatrix},$$

it follows that for $\sigma \neq \lambda_i(u^-)$ there exists a curve $s \mapsto \sigma(s), u^+(s)$, whose derivative is $(1, 0) \in \mathbb{R}^\sigma \times \mathbb{R}^n$: clearly this curve coincides with $u^+ = u^-$, $\sigma + s$.

If instead we consider the point $(\lambda_i(u^-), u^-)$, then we obtain a smooth surface, which can be parametrized by $\sigma, s = -u^- + \ell_i(u^-) \cdot u^+ = u_i^+ - u_i^-$:

$$u^+ = u^+(\sigma, s), \quad D_{\sigma, u_i^+} u \llcorner_{(\lambda_i(u^-), u^-)} = (0, r_i(u^-)).$$

Substituting

$$-\sigma(u^+(\sigma, s) - u^-) + f(u(\sigma, s)) - f(u^-) = 0.$$

Differentiation w.r.t. σ, s we get

$$-(u^+ - u^-) - \sigma u^+_\sigma + Df(u^+) u^+_\sigma = 0,$$
$$-u^+_s - \sigma u^+_{s\sigma} + Df(u^+) u^+_{s\sigma} + D^2 f(u^+) : u^+_s \otimes u^+_\sigma = 0.$$

For $s = 0, \sigma = \lambda_i(u^-)$ we obtain

$$u_\sigma = 0, \quad u_{\sigma s} = 0,$$

where we used that $u_i = s$.

Projecting on the i-th component we obtain

$$-\sigma s + f_i(u(s, \sigma)) - f_i(u^-) = 0 \quad \Rightarrow \quad \sigma = \frac{f_i(u(s, \sigma)) - f_i(u^-)}{s} = g_i(s, \sigma),$$

which gives $\sigma = \sigma(s)$: indeed by

$$\lim_{s \to 0} \frac{f_i(u(s, \sigma)) - f_i(u^-)}{s} = Df_i(u^-) r_i(u^-) = \lambda_i(u^-),$$

and $u_{s\sigma} = 0$ gives

$$\partial_\sigma g_i(s, \sigma) \llcorner_{s=0} = \lim_{s \to 0} \frac{Df_i(u(s, \sigma)) \partial_\sigma u}{s} = Df_i(u^-) \partial_{s\sigma} u = 0.$$

Thus

$$\sigma - g_i(s, \sigma) = 0$$

is invertible at $\sigma = \lambda_i(u^-)$, giving a curve $s \mapsto \sigma(s)$ such that

$$\begin{aligned} \sigma_s \llcorner_{s=0} &= \partial_s g_i(0, \lambda_i(u^-)) \llcorner_{s=0} = \lim_{s \to 0} \frac{f(u(s, \sigma)) - f_i(u^-) - s Df_i(u(s, \sigma)) u_s}{s^2} \\ &= \frac{1}{2} D^2 f_i : r_i \otimes r_i = \frac{1}{2} D\lambda_i r_i. \end{aligned}$$

In the g.n.l. case it is invertible.

The second derivative of the curve $s \mapsto u(s) = u(s, \sigma(s))$ is then obtained by

$$-\sigma_s(u - u^-) - \sigma u_s + Df u_s = 0 \quad \Rightarrow \quad u_s(0) = r_i(u^-),$$
$$-\sigma_{ss}(u - u^-) - 2\sigma_s u_s - \sigma u_{ss} + D^2 f : u_s \times u_s + Df u_{ss} = 0,$$
$$(Df - \lambda_i \mathrm{id}) u_{ss} = -D^2 f : r_i \otimes r_i + (D\lambda_i r_i) r_i = \sum_j (\lambda_j - \lambda_i) r_j \langle \ell_j, Dr_i r_i \rangle,$$

where we have used

$$0 = D\langle \ell_j, r_i \rangle = \langle D\ell_j r_i, r_i \rangle + \langle \ell_j, Dr_i r_i \rangle.$$

Hence, taking into account that $Dr_i r_i$ is orthogonal to r_i, we conclude that

$$u_{ss} = Dr_i r_i.$$

This is the same second derivative of the rarefaction curve.

The *entropy admissibility* is to require that $s \leq 0$, i.e. the eigenvalue is decreasing along the shock curve, or equivalently

$$\lambda_i(u^-) < \sigma < \lambda_i(u^+) \quad \text{if } s < 0.$$

5.1. Construction of admissible waves and solution to the Riemann problem. The solution to the Riemann problem is then constructed as follows.

(1) The admissible curves $T_i(s, u^-)$ are given by joining together the rarefaction curves $R_i(s, u^-)$ for $s \geq 0$ with the shock curves $S_i(s, u^-)$ for $s < 0$. The previous estimates shows that these curves are $C^{2,1}$ (their first and second derivative coincide), with derivative $r_i(u^-)$ at $s = 0$. Each side of the curve defines an elementary wave, either rarefaction or shock.

(2) Define the map

$$(s_1, \dots, s_n) \mapsto T(s_1, \dots, s_n, u^-) = T_n(s_n) \circ T_{n-1}(s_{n-1}) \circ \cdots \circ T_1(s_1, u^-) = u^+. \tag{5.2}$$

This maps defines a solution of the Riemann problem with u^- on the left hand side and u^+ on the right hand side by patching together the elementary waves

$$T_i(s_i) \circ \cdots \circ T_1(s_1, u^-) = u_i^+ = u_{i+1}^-+, \quad T_{i+1}(s_{i+1}) \circ \cdots \circ T_1(s_1, u^-) = u_{i+1}^+ = u_{i+2}^-.$$

(3) It remains to show that given u^+ we can find the coordinates $(s_1, \dots, s_n)$. The map (5.2) is C^2 with derivative

$$D_s T \llcorner_{s=0} = \begin{bmatrix} r_1(u^-) & \cdots & r_n(u^-) \end{bmatrix},$$

which is invertible: hence there is a local neighborhood such that there exists a unique solution.

Remark 5.2. We are not saying anything about the entropy, because in general it is possible that no entropy exists. The stability condition is at the level of characteristics: each jump has the characteristics of the same family entering on both sides.

5.2. Exercises.

(1) Consider the p-system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho, e)) = 0, \\ (\rho e)_t + (\rho e u + p(\rho, e) u) = 0, \end{cases} \qquad p(\rho, e) = \rho e.$$

Prove that the system is strictly hyperbolic.

(2) Compute the eigenvalue, left/right eigenvectors and prove that it has two gnl characteristic speeds and one linearly degenerate,

(3) Construct the shock curves and the rarefaction curves.

(4) Repeat the analysis for the isentropic case

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho)) = 0, \end{cases} \qquad p(\rho) = \rho^\gamma, \ \gamma > 1.$$

(5) Repeat the analysis for the isentropic case in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(\frac{1}{v})_x = 0, \end{cases} \qquad p\left(\frac{1}{v}\right) = v^{-\gamma}, \ \gamma \geq 1.$$

(6) Show that the two solutions are equivalent even if discontinuous: recall that the Lagrangian coordinates (t, y) are given by

$$x = X(t, y), \quad \dot{X} = u(t, X), \ X(0, y) = y.$$

6. Interaction estimates and Glimm functional

The construction of a solution is based on the interaction estimates among waves (shocks and rarefaction), in a similar way as in the scalar case.

6.1. Approximate solutions. There are various schemes to construct approximate solution: their main characteristic is that the solution evolves as the solution to several Riemann problems, and then when the waves interact the scheme constructs an approximate interaction in order to control the number/location of waves. The main problem to avoid is accumulation points of interactions, as in the scalar case.

The *Glimm scheme* constructs approximate solution by joining nearby Riemann problem with a probabilistic parameter, and shows that if the parameter distribution is uniform then the limit up to subsequences is a solution of the PDE.

The *Wave-front tracking* approximates the solution to the Riemann problem in order to control both the number of waves and the number of interaction.

The main tool in both cases is the construction of a decreasing functional, bounding the TV of u: it is called the *Glimm interaction functional.*

6.2. Glimm interaction functional. We construct the Glimm interaction functional in the most general case.

6.2.1. *Interaction estimates.* Consider two subsequent Riemann problems,

$$[u^-, u^m],\ [u^m, u^+] \quad \text{and their merge } [u^-, u^+].$$

The idea is to study how different are the solution before the interaction and after the interaction. In other words, if

$$u^m = T_n(s'_n) \circ \cdots \circ T_1(s'_1)u^-, \quad u^+ = T_n(s''_n) \circ \cdots \circ T_1(s''_1)u^m, , \quad u^+ = T_n(s_n) \circ \cdots \circ T_1(s_1)u^-,$$

we need to compute

$$\sum_i \big| s_i - (s'_i + s''_i) \big|,$$

which is a measure of how much the total variation of u changes after the interaction. Being T smooth, it is enough to see

$$\mathcal{I} = \big| T_n(s'_n + s''_n) \circ \cdots \circ T_1(s'_1 + s''_1)u^- - T_n(s''_n) \circ \cdots \circ T_1(s''_1) \circ (T_n(s'_n) \circ \cdots \circ T_1(s'_1)u^-) \big|.$$

First, if there exists k such that

$$\begin{cases} s'_i = 0 & \text{if } i > k, \\ s'_k, s''_k > 0 & \text{for } i = k, \\ s''_i = 0 & \text{if } i < k, \end{cases}$$

then $\mathcal{I} = 0$ because of the definition of T and the semigroup property of ODE applied to the rarefaction curves. Another way of seeing this is that the waves of the two original Riemann problems are not approaching, so that when we put them together there is no interaction.

Using the Lipschitz property we then have

$$\mathcal{I} \leq \mathcal{O}(1) \bigg(\sum_{i<j} |s''_i||s''_j| + \sum_k \big(|s'_k|[s''_k]^- + [s'_k]^- |s''_k| \big) \bigg).$$

The first term is called the *transversal term*: it is related to the interaction among different families. The sum is on the waves fo the first Riemann problem which are faster than the waves of the second Riemann problem, so that they will overtake them. The second terms are related to the interaction among the same family: it is due to the fact that shocks and shocks or rarefactions are going to interact, because of the fact that characteristics are entering the shock.

6.2.2. *Glimm interaction functional.* The idea is to construct a functional decreasing when we join two Riemann problem together.

Let u be a piecewise constant function, with jumps

$$[u_a^-, u_a^+], \quad u_a^+ = T_n(s_{n,a}) \circ \cdots \circ T_1(s_{1,a}) u_a^-.$$

Using the form of $\mathcal{I}$, we define the *Glimm functional* as

$$\mathcal{Q}(u) = \sum_{a<a'} \sum_{i<j} |s_{j,a}||s_{i,a'}| + \sum_{a,a'} \sum_k [s_{k,a}]^- |s_{k,a'}|.$$

The first part is the transversal interaction: it counts the couple of waves where the faster one is before the slower, so that they are going to cross in the future.

The second part counts all the couples of the same family where one is a shock, so that they are going to interact in the future.

6.3. Glimm functional and control of the BV norm. The Glimm functional is now

$$\mathcal{V}(u) = \text{Tot.Var.}(u) + \mathcal{Q}(u) = \sum_{a,i} |s_{i,a}| + C\mathcal{Q}(u).$$

The next proposition is the fundamental result in this direction: it states that when interaction occurs, $\mathcal{V}$ is decreasing. This is used both for the control of the BV norm and for the control of the number of waves, in a similar fashion as in the scalar case (where Tot.Var. is decreasing naturally).

PROPOSITION 6.1. *If we replace two consecutive Riemann problems with their union, let us say* $[u_{-1}, u_0]$, $[u_0, u_1]$ *with* $[u_{-1}, u_1]$, *then* $\mathcal{V}$ *decreases if the total variation is sufficiently small.*

PROOF. It is enough to prove that $\mathcal{Q}$ decreases of a quantity equivalent to the increase of $\mathcal{I}$ up to a constant, which is the C in the definition of $\mathcal{V}$.

After the interaction, all the couples of waves in $\mathcal{Q}$ belonging to the Riemann problems $[u_{-1}, u_0]$, $[u_0, u_1]$ disappears, so that a negative term is exactly

$$\sum_{i<j} |s_{i,0}||s_{j,-1}| + \sum_k \big(|s_{k,0}|[s_{k,1}]^- + [s_{k,0}]^- |s_{k,1}|\big).$$

The increase is due to the fact that all other terms changes according to

$$\mathcal{O}(1) \sum_a \sum_{i,j} \big(|s'_{i,0}| - |s_{i,0}| - |s_{i,1}|\big)|s_{j,a}|,$$

because the new amount of waves of the i-th family is not exactly the sum of the two previous ones.

Then we have

$$\begin{aligned} \Delta\mathcal{Q} &= -(1 + C\text{Tot.Var.}(u)) \bigg(\sum_{i<j} |s_{i,0}||s_{j,-1}| + \sum_k \big(|s_{k,0}|[s_{k,1}]^- + [s_{k,0}]^- |s_{k,1}|\big) \bigg) \\ &\leq \frac{1}{2} \bigg(\sum_{i<j} |s_{i,0}||s_{j,-1}| + \sum_k \big(|s_{k,0}|[s_{k,1}]^- + [s_{k,0}]^- |s_{k,1}|\big) \bigg), \end{aligned}$$

if Tot.Var.$(u) \ll 1$. □

COROLLARY 6.2. *The domain*

$$\big\{u : \mathcal{V}(u) \leq \epsilon\big\}$$

is closed for L^1*-convergence and invariant for the flow of the PDE.*

PROOF. The invariance for the flow is in the sense that approximate solutions remains inside this domain.

The closure follows if we show that Tot.Var. and $\mathcal{Q}$ are l.s.c. w.r.t. L^1-convergence. Since we did not give the continuous formula for $\mathcal{V}$, we will only prove the statement for piecewise constant functions: this is just Proposition 6.1. □