UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dipartimento di Fisica

Dipartimento d'Eccellenza 2023-2027

993SM – Laboratory of Computational Physics Unit XI December 5, 2025

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The logistic map

fixed-point condition is given by $f(x^*) = x^*$

$$x_{n+1} = 4rx_n(1 - x_n)$$

$$\Rightarrow \quad x^*_1 = 0 \qquad \text{and} \qquad x^*_2 = 1 - \frac{1}{4r}$$

stable fixed point :

the iterated values of x converge to x^* independently of the value of x_0

unstable fixed point :

if for almost all x_0 near the fixed point, the trajectories diverge from it

It can be demonstrated that:

$$x^*_1 = 0 \text{ is stable for } 0 < r < 1/4$$

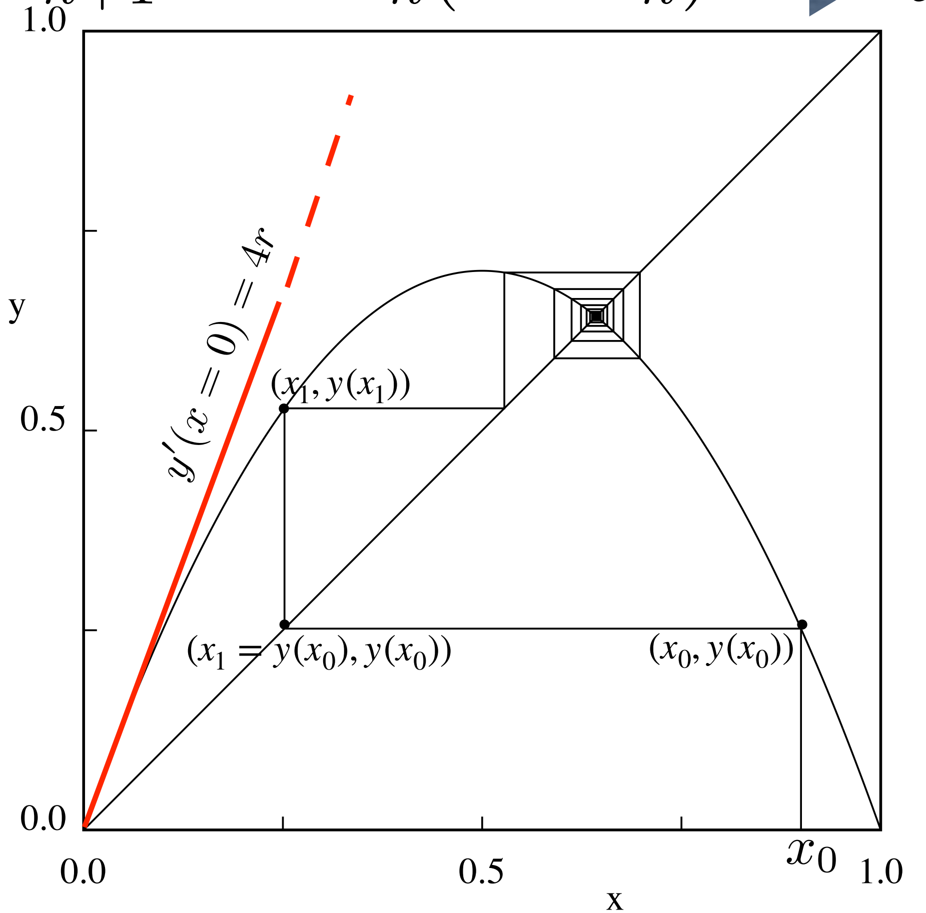
$$x^*_2 = 1 - \frac{1}{4r} \quad \text{is stable for} \quad \frac{1}{4} < r < \ldots ? \qquad (< 1)$$

(condition $x^*_2 > 0$)

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \rightarrow y(x) = 4rx(1 - x)$$

$$y'(x = 0) = 4r$$



$$r = 0.7$$
$$x_0 = 0.9$$

Graphical representation of the iteration of the logistic map (cobweb plot) : the graphical solution converges to the fixed point $x^* \approx 0.643$

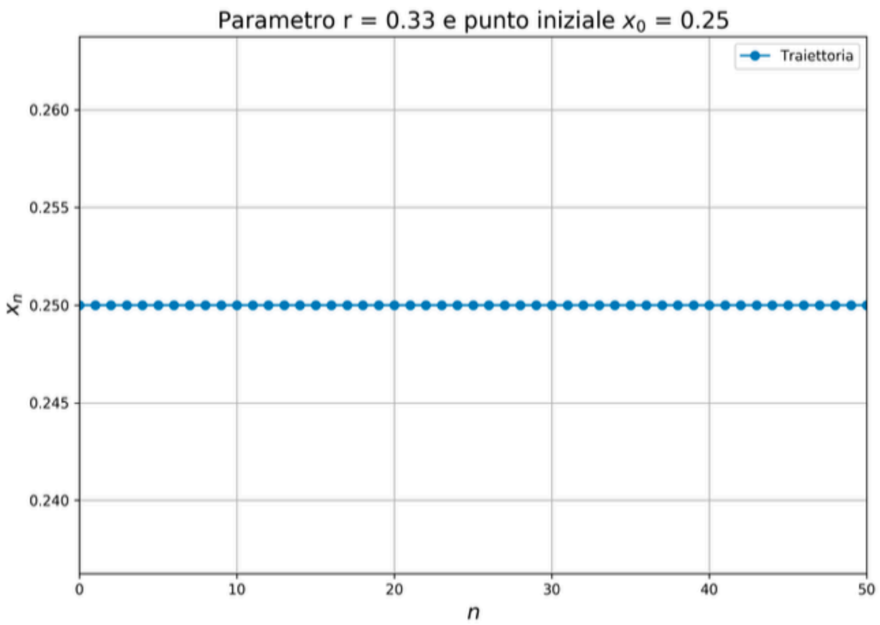
Note: the graphical intersection between y(x) and the diagonal gives the **fixed point**, but it is not sufficient to determine whether it is **stable or unstable**

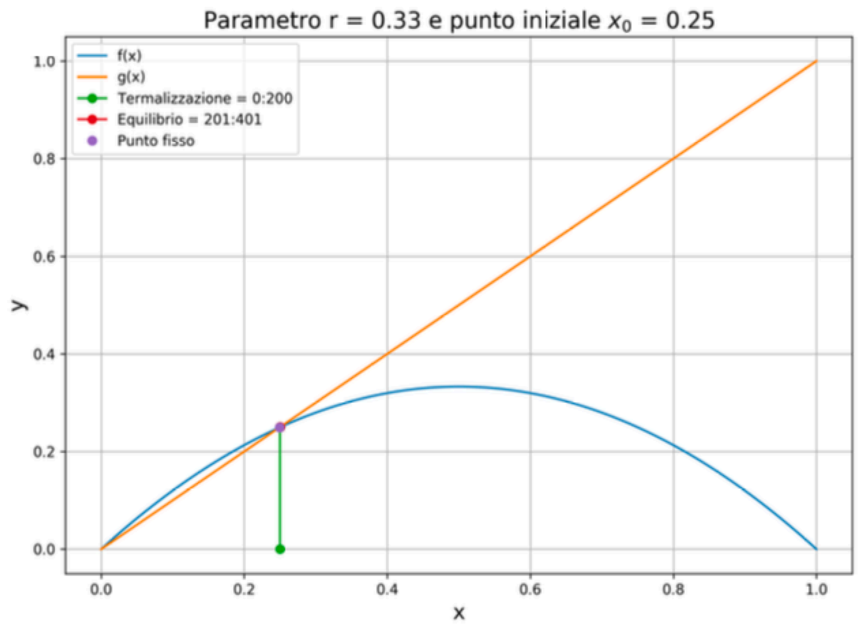
The logistic map

$$x_{n+1} = 4rx_n(1 - x_n)$$

$$r = 0.33 \Longrightarrow x^* = 0.25$$

$$x_0 = 0.25$$





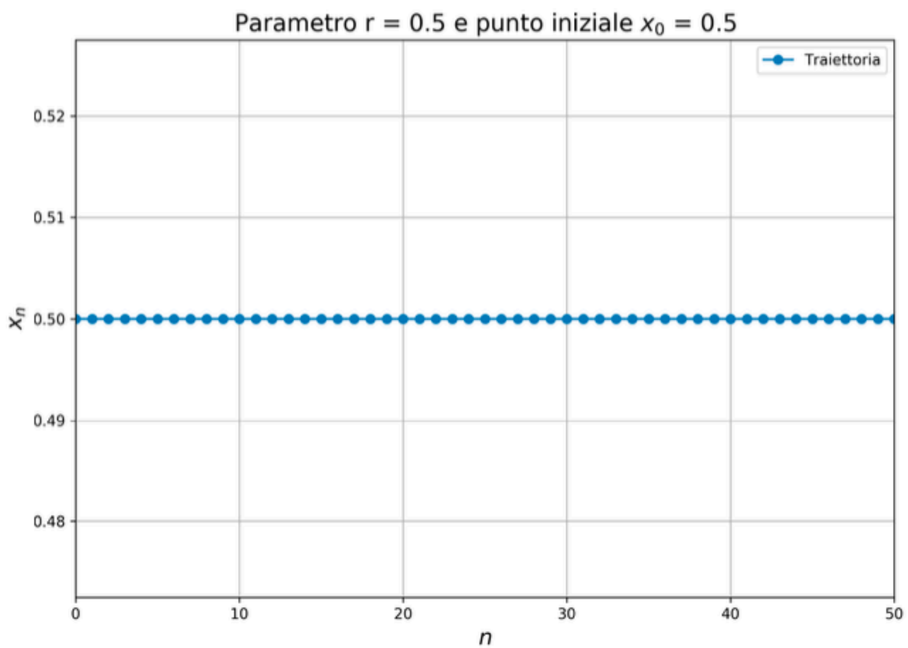
stable fixed point

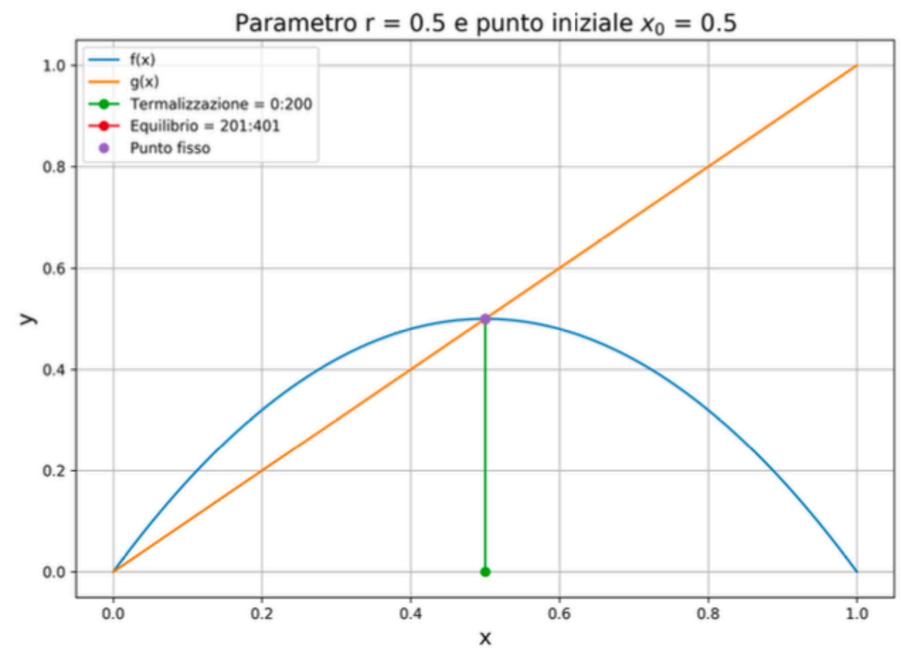
by Roberta Favata

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.5 \Longrightarrow x^* = 0.5$$

$$x_0 = 0.5$$





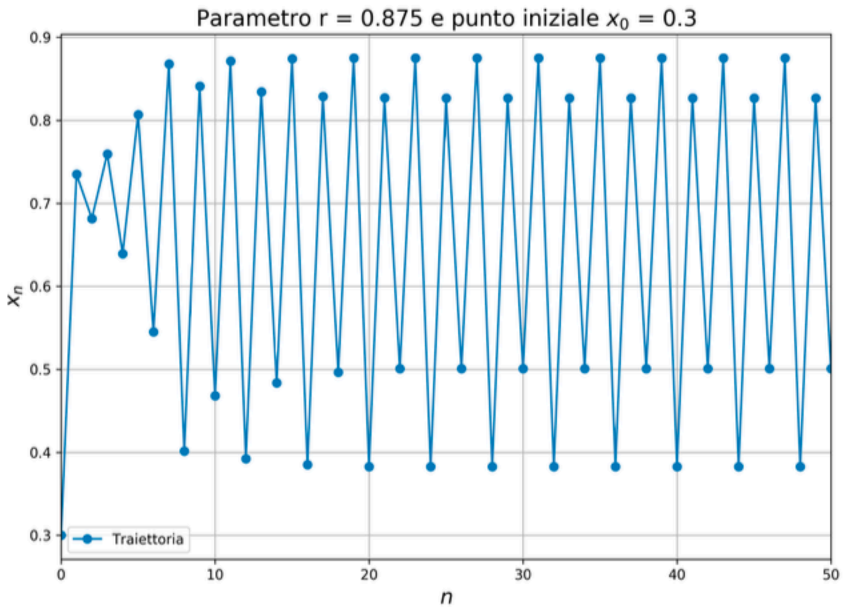
stable fixed point

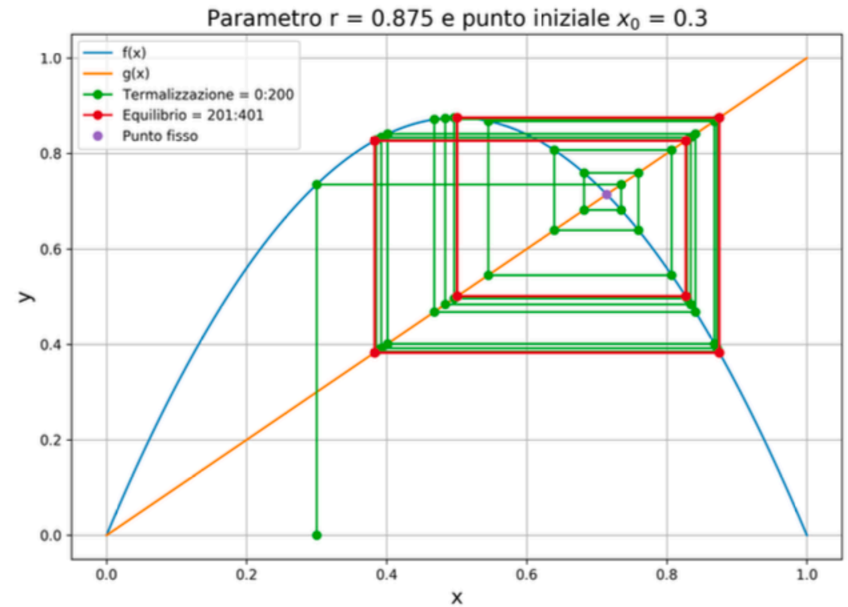
by Roberta Favata

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.875 \Longrightarrow \quad x^* = 1 - \frac{1}{4r} \approx 0.714285714.$$

$$x_0 = 0.3$$





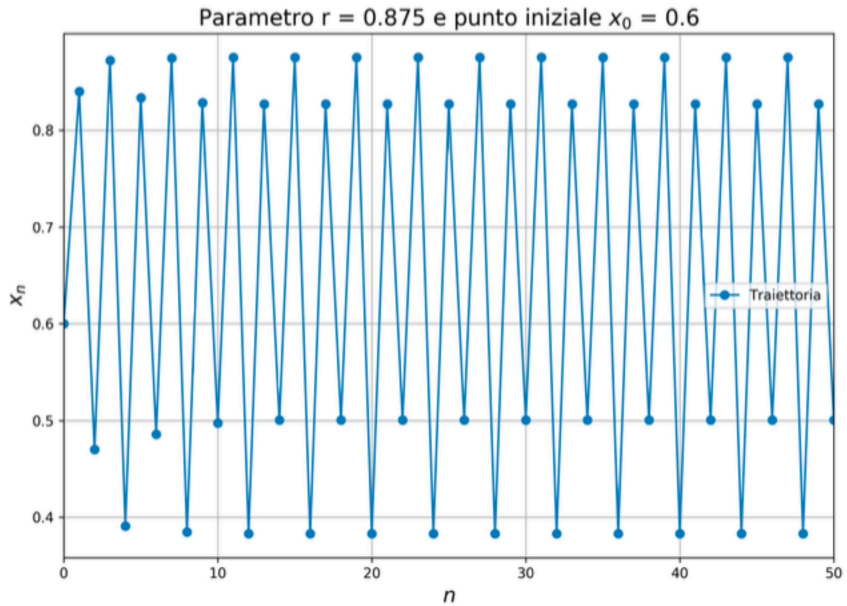
$$x_1 = 0.36282 \qquad x_2 = 0.50088 \qquad x_3 = 0.82694 \qquad x_4 = 0.875$$

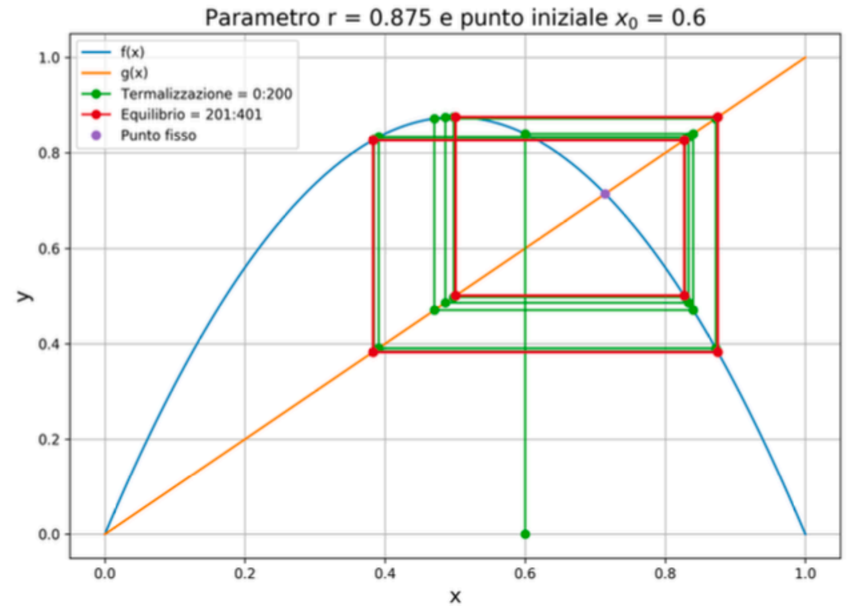
by Roberta Favata

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.875 \Longrightarrow \quad x^* = 1 - \frac{1}{4r} \approx 0.714285714.$$

$$x_0 = 0.6$$





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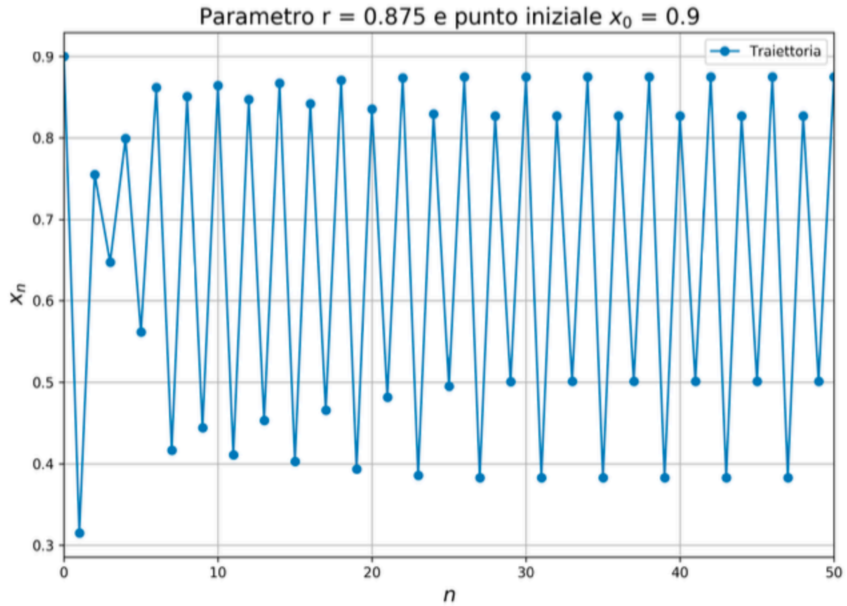
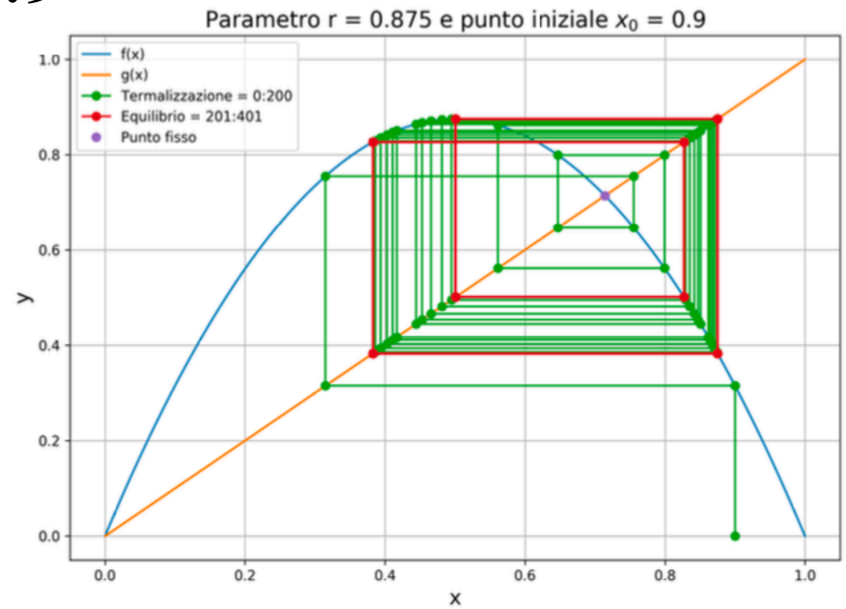
The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.875 \Longrightarrow \qquad x^* = 1 - \frac{1}{4r} \approx 0.714285714.$$

$$x_0 = 0.9$$





$$x_1 = 0.36282 \qquad x_2 = 0.50088 \qquad x_3 = 0.82694 \qquad x_4 = 0.875$$

by Roberta Favata

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n)$$

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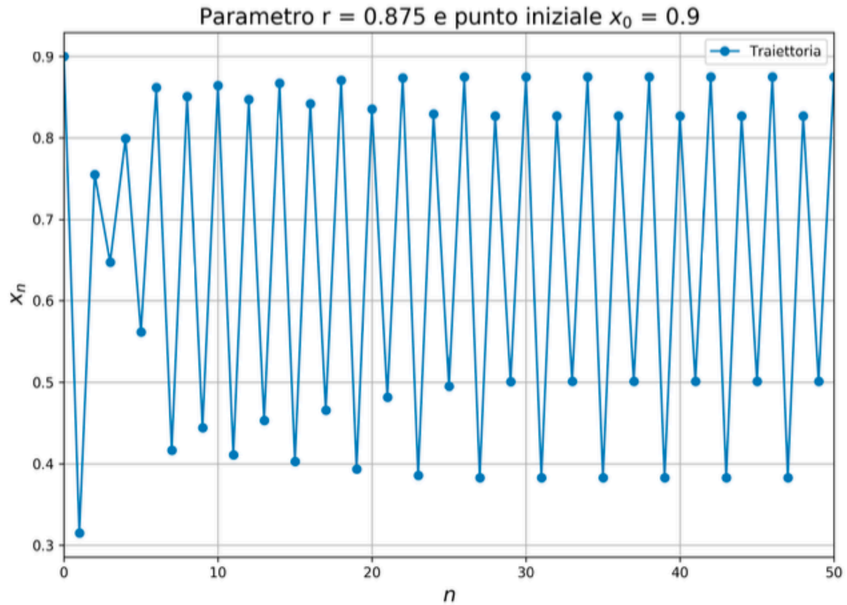
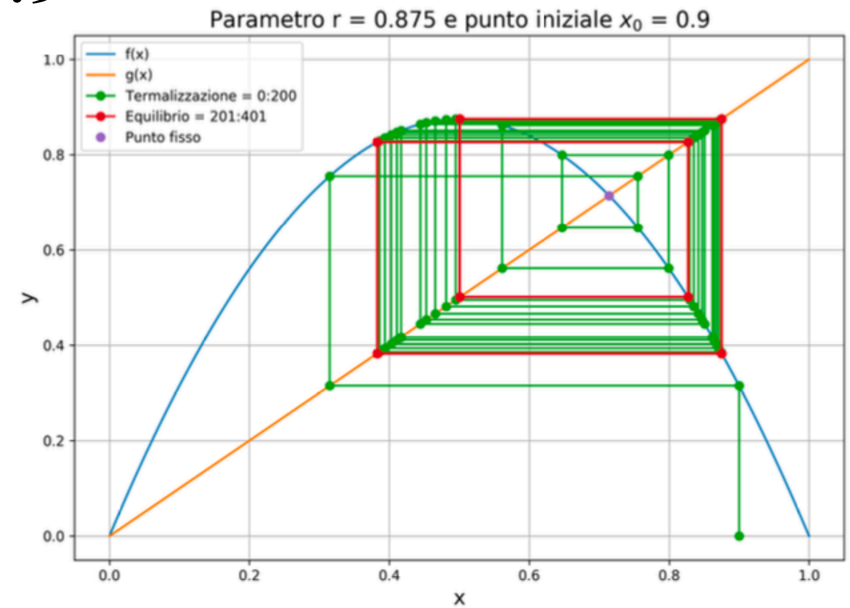




$$x_1 = 0.36282 \qquad x_2 = 0.50088 \qquad x_3 = 0.82694 \qquad x_4 = 0.875$$

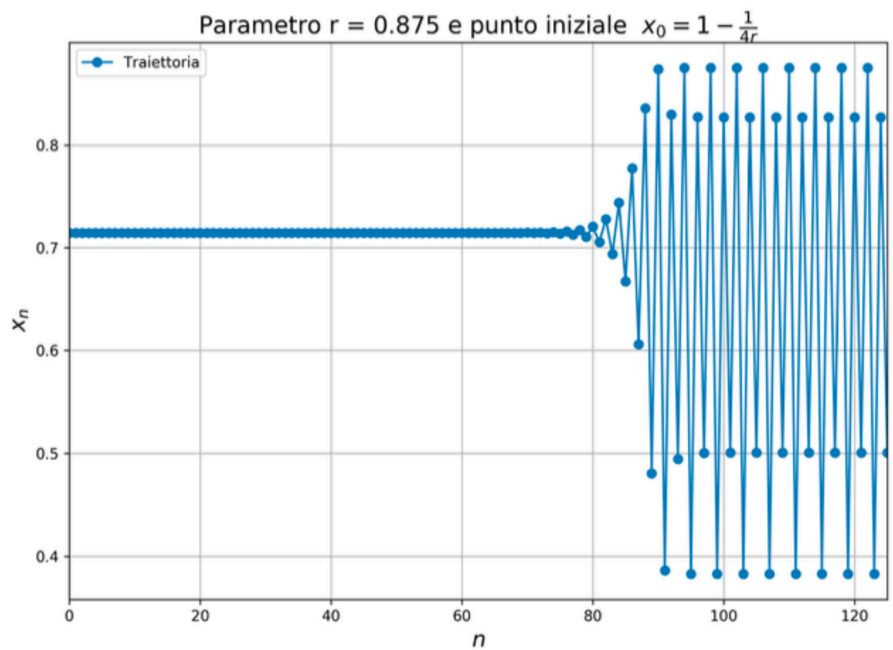
by Roberta Favata

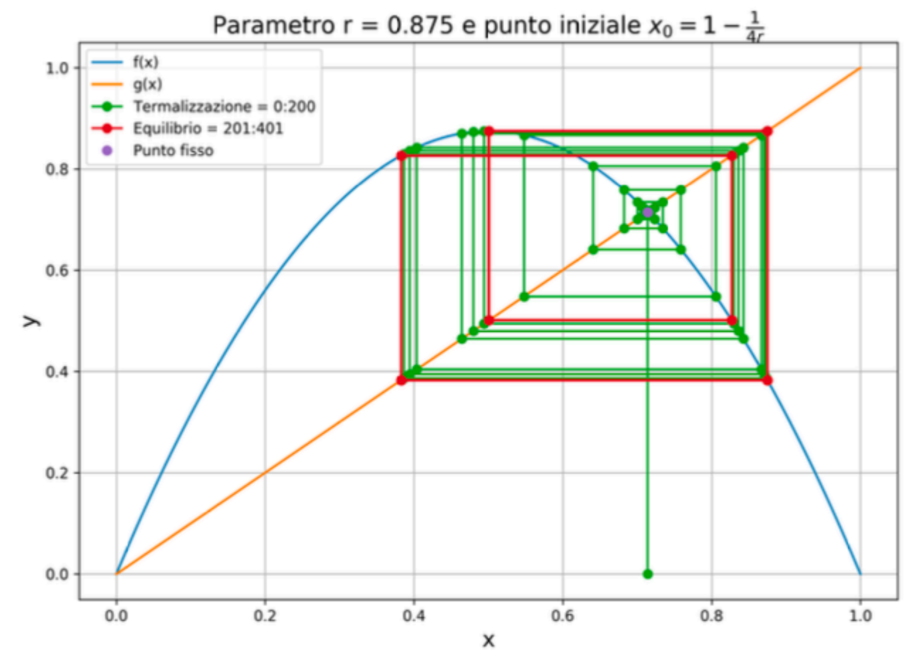
The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.875 \implies \quad x^* = 1 - \frac{1}{4r} \approx 0.714285714.$$

$$x_0 = 0.714285714$$

(machine precision)





unstable fixed point !!!

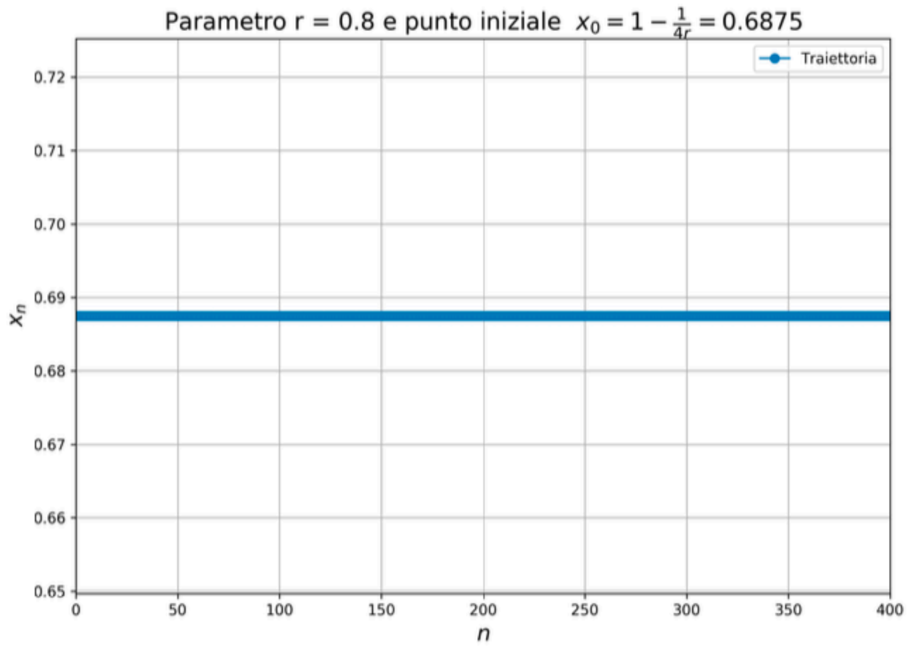
by Roberta Favata

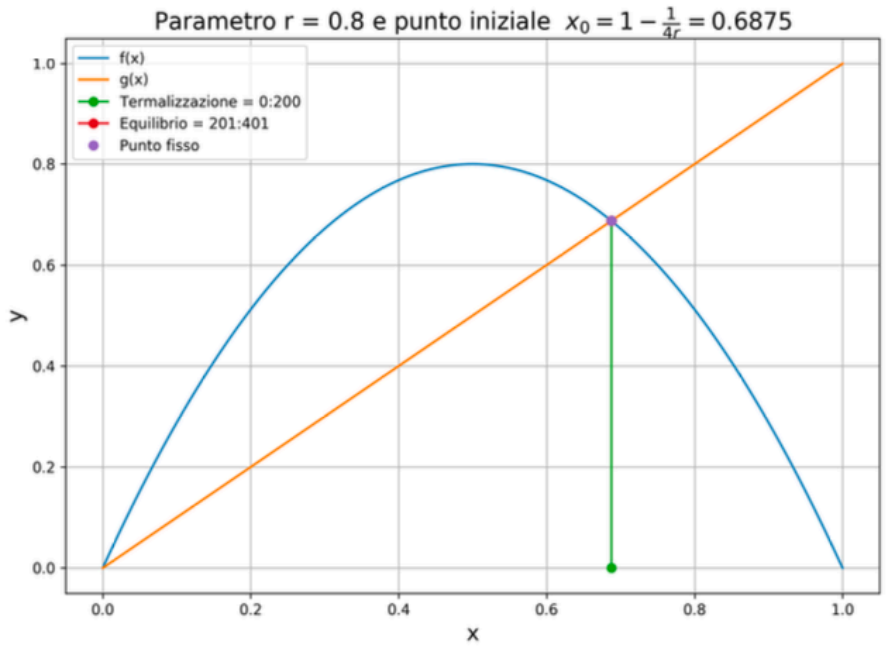
The logistic map

$$x_{n+1} = 4rx_n(1 - x_n)$$

$$r = 0.8 \Longrightarrow x^* = 1 - \frac{1}{4r} = 0.6875$$

$$x_0 = 0.6875$$





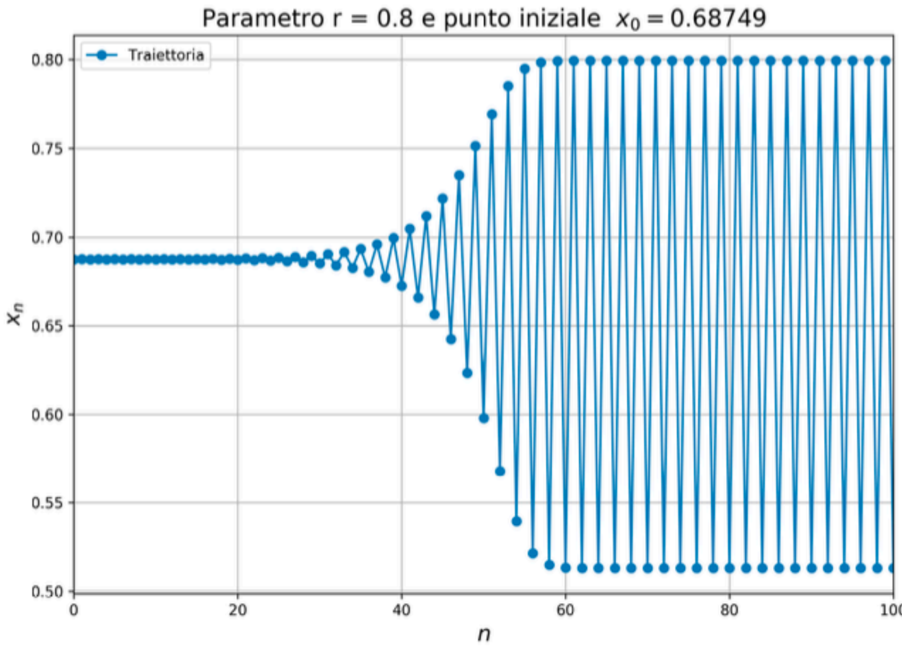
it seems stable…. BUT….

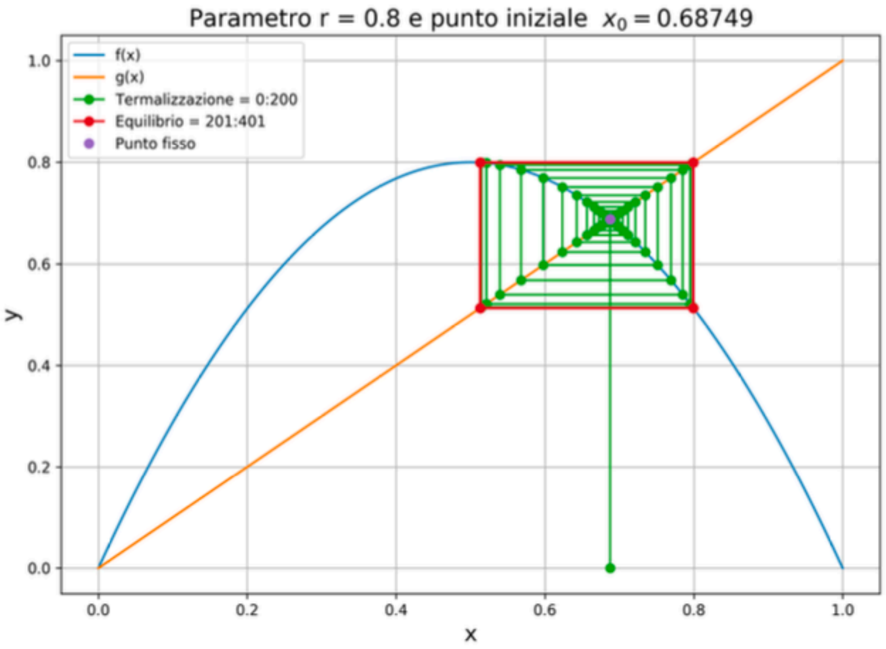
by Roberta Favata

The logistic map

$$x_{n+1} = 4rx_n(1 - x_n) \qquad r = 0.8 \Longrightarrow x^* = 1 - \frac{1}{4r} = 0.6875$$

$$x_0 = 0.687\mathbf{49}$$





unstable fixed point!

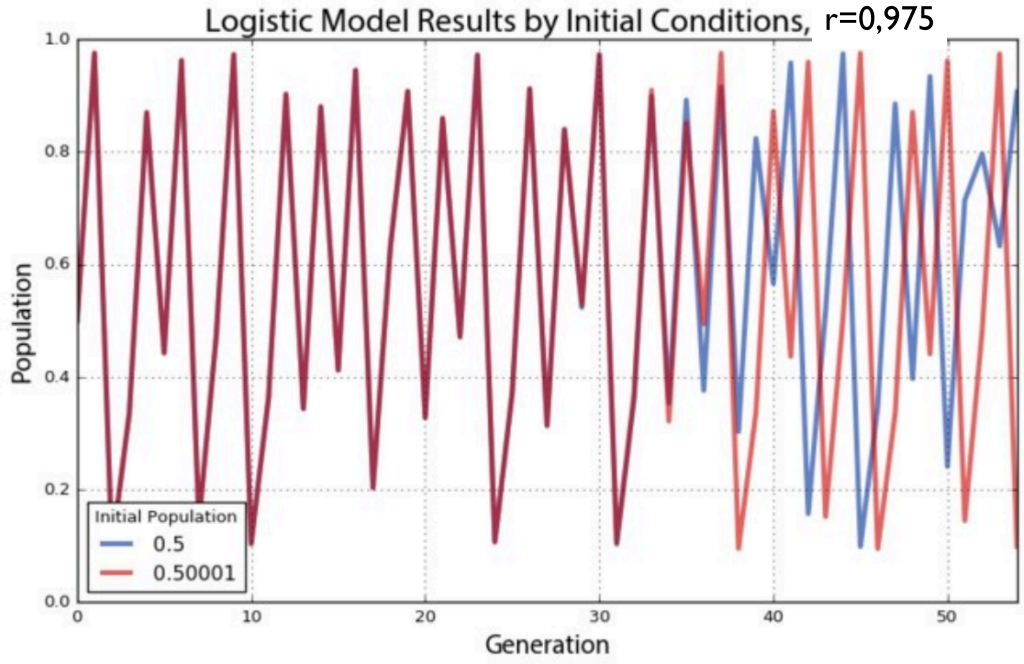
by Roberta Favata

Measuring chaos

$\Delta x_0 = 0.00001, \quad \Delta x_{n>40} = ???$

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}$$

Lyapunov exponent

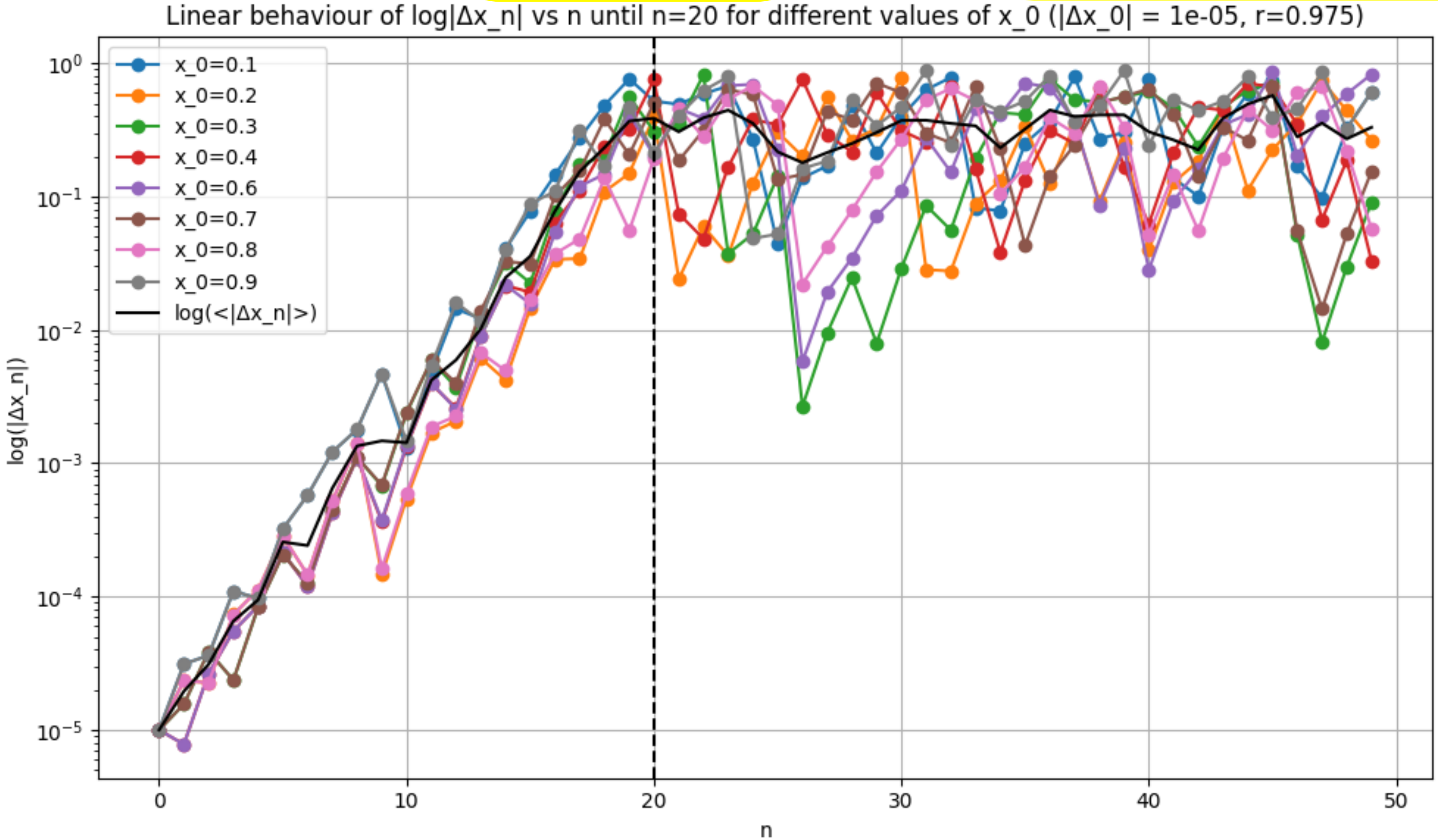


The difference between two trajectories may diverge exponentially,

but not indefinitely since the trajectory is limited!

Measuring chaos

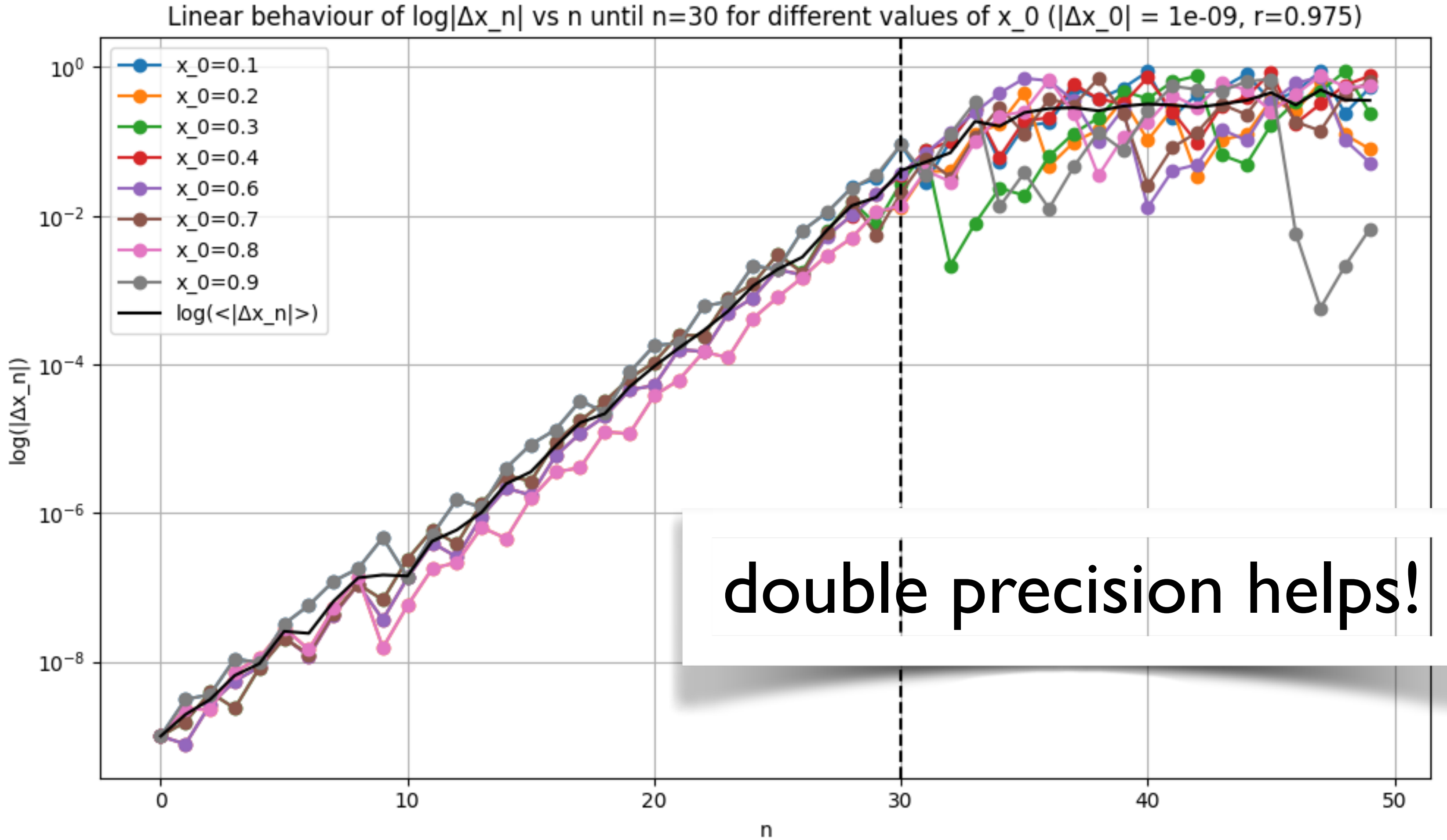
$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n} \Rightarrow \ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \lambda n \quad \text{or} \quad \ln|\Delta x_n| = \lambda n + \ln|\Delta x_0|$$



by Elena Fornasier

Measuring chaos

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n} \Rightarrow \ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \lambda n$$ or $$\ln|\Delta x_n| = \lambda n + \ln|\Delta x_0|$$

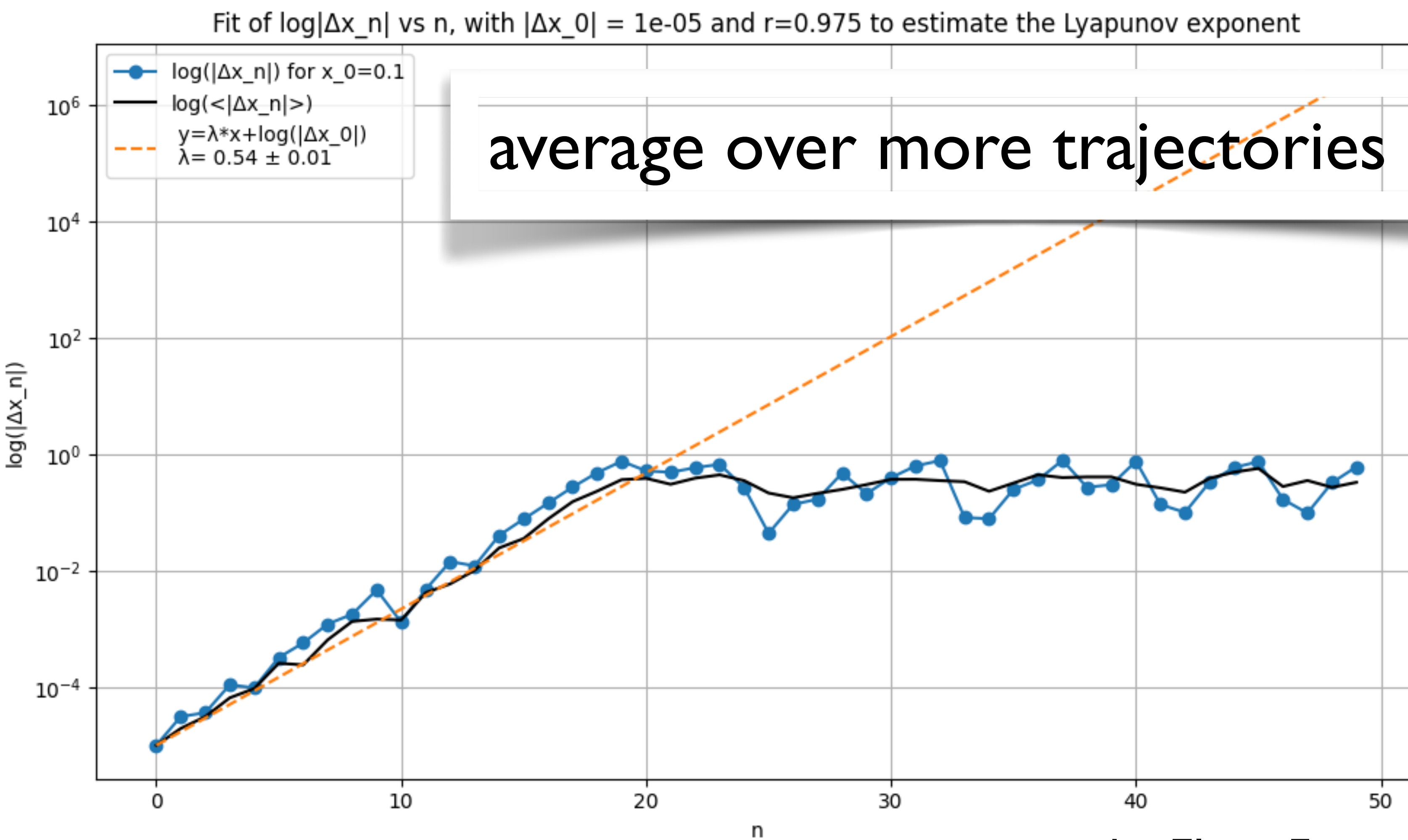


double precision helps!

by Elena Fornasier

Measuring chaos

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n} \Rightarrow \ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \lambda n \quad \text{or} \quad \ln|\Delta x_n| = \lambda n + \ln|\Delta x_0|$$

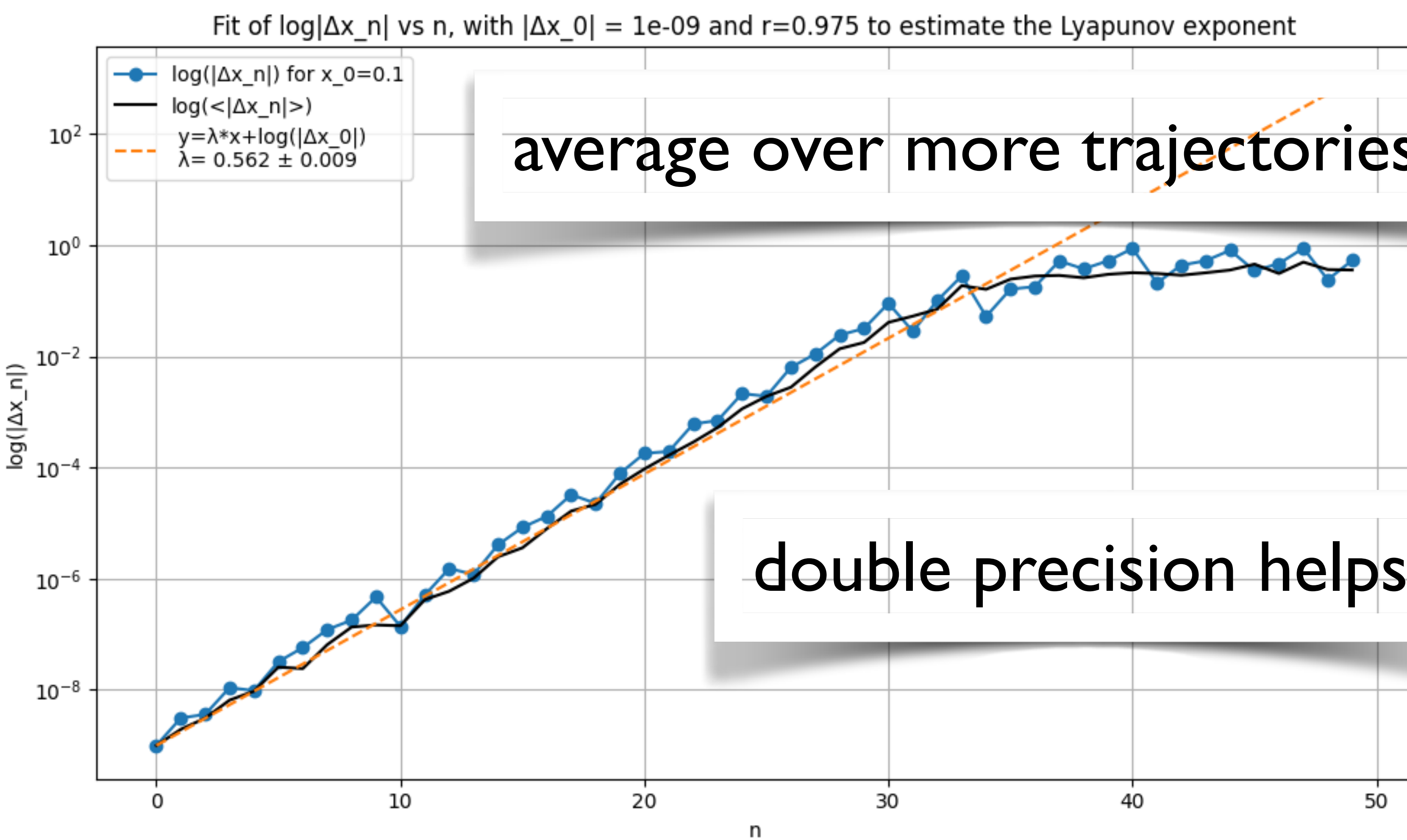


average over more trajectories helps!

by Elena Fornasier

Measuring chaos

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n} \Rightarrow \ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \lambda n \quad \text{or} \quad \ln|\Delta x_n| = \lambda n + \ln|\Delta x_0|$$



average over more trajectories helps!

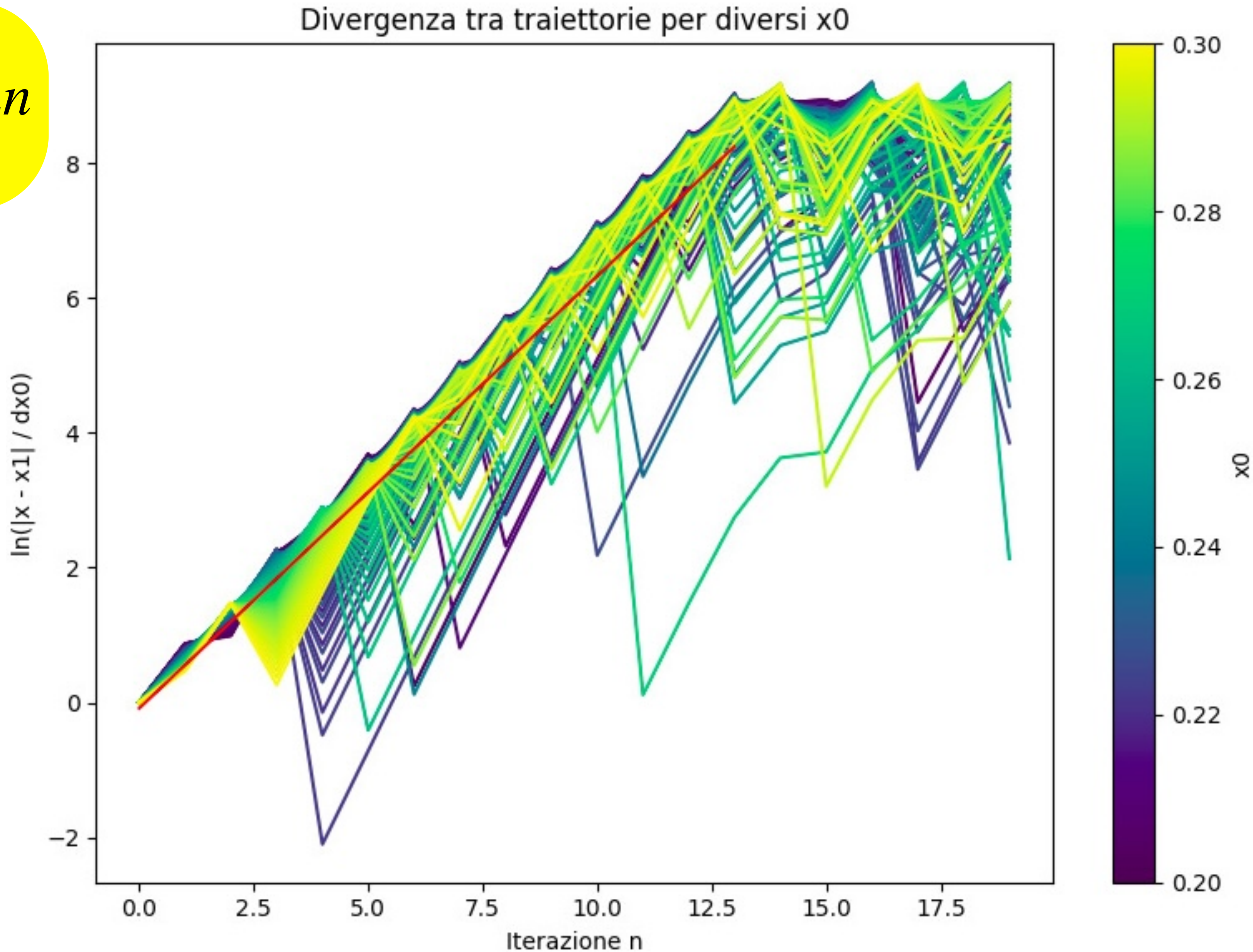
double precision helps!

by Elena Fornasier

Measuring chaos

$$\ln \left| \frac{\Delta x_n}{\Delta x_0} \right| = \lambda n$$

dx0 = 0.001
step = 0.0001

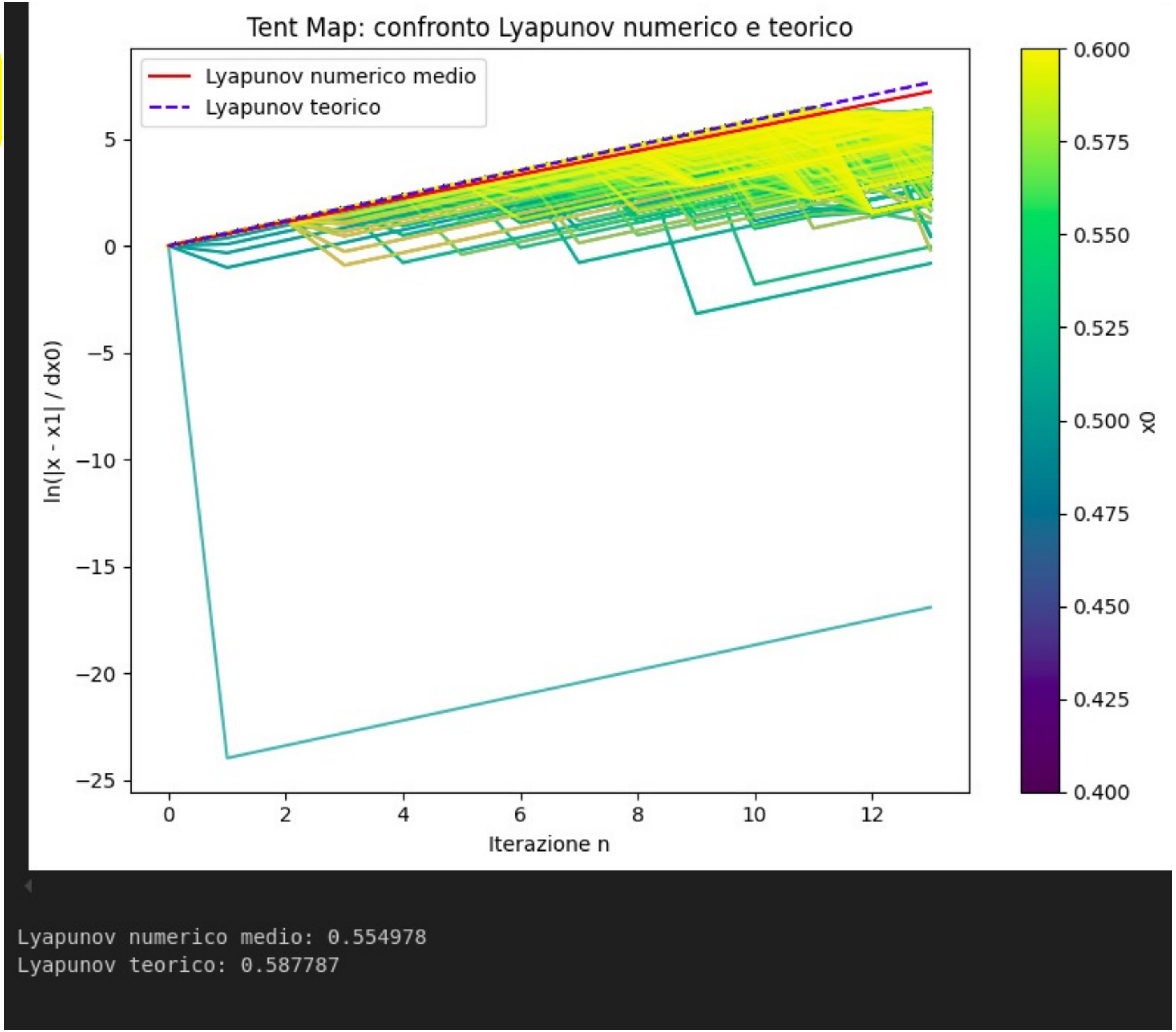


by Matteo Falasca

Measuring chaos

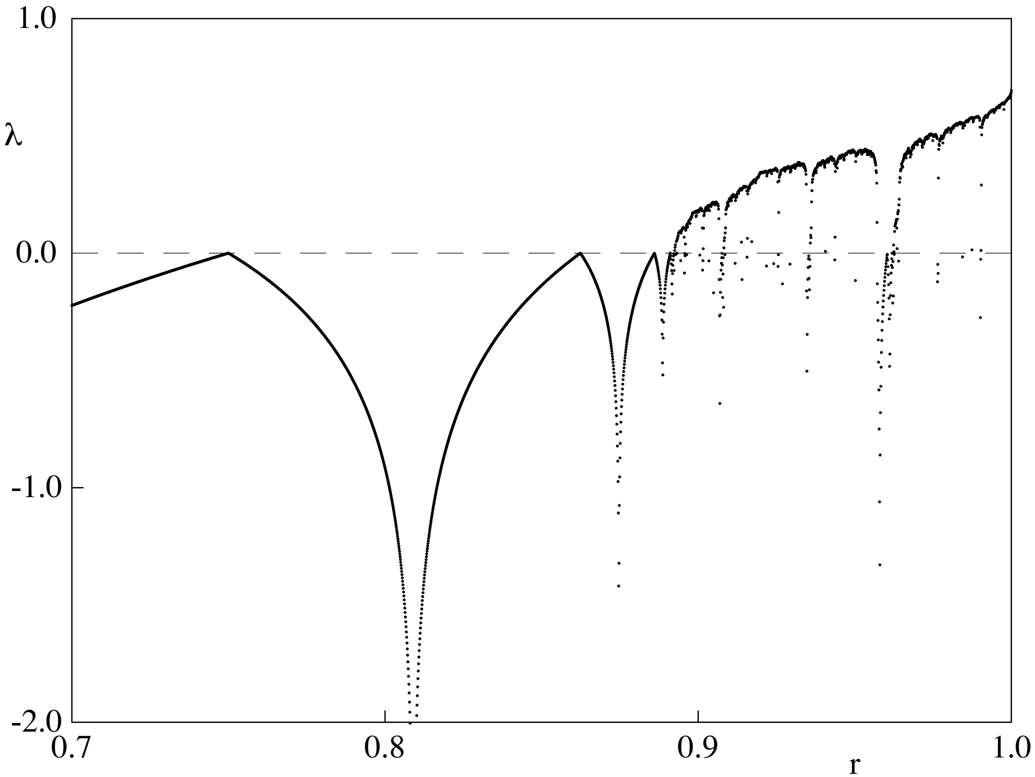
$$\ln \left| \frac{\Delta x_n}{\Delta x_0} \right| = \lambda n$$

altro range di x0



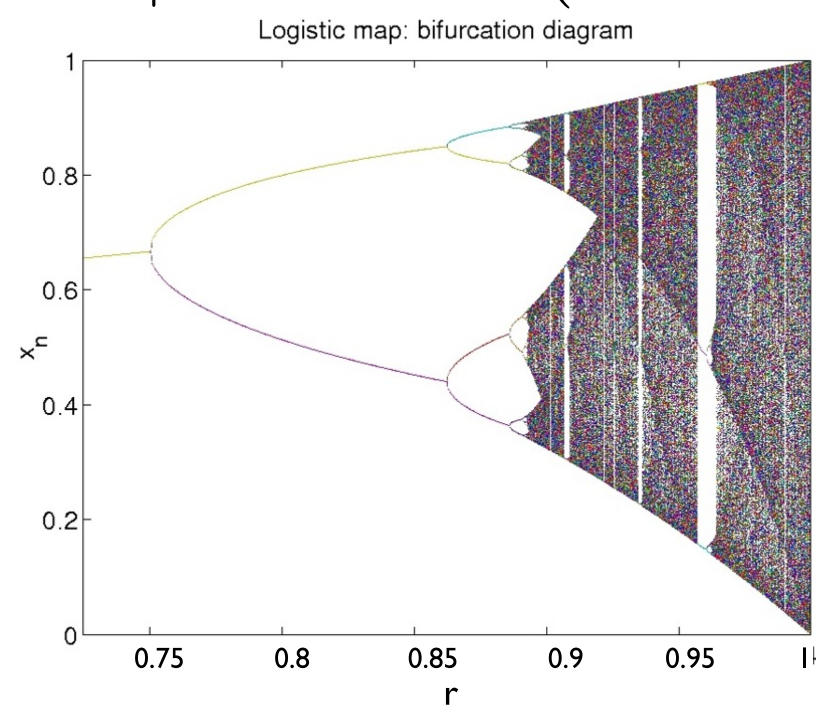
by Matteo Falasca

Measuring chaos



The Lyapunov exponent as a function of the control parameter r for the logistic map

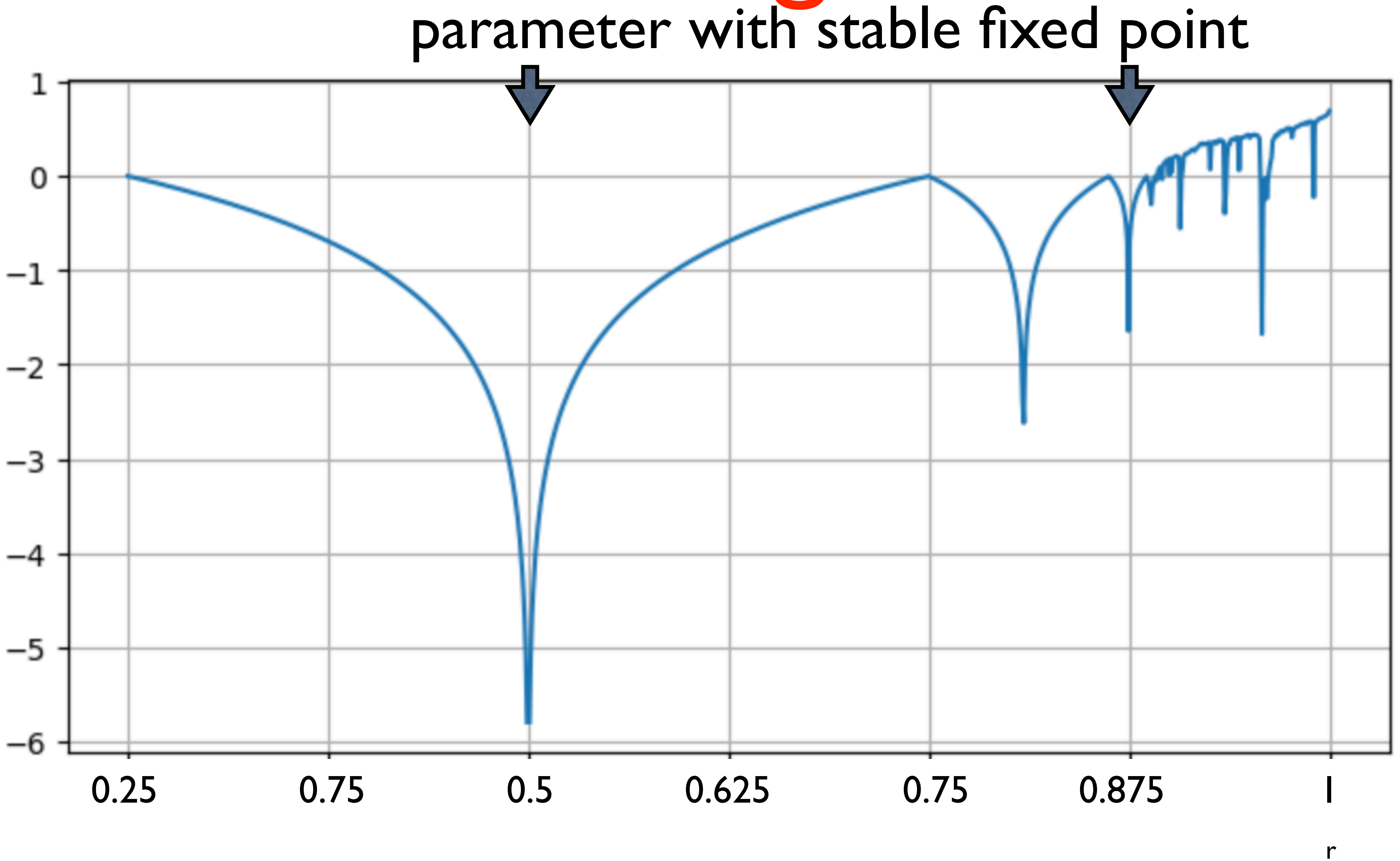
$$x_{n+1} = 4rx_n(1 - x_n)$$



$$|\Delta x_n| = |\Delta x_0| \, e^{\lambda n}$$

Lyapunov exponent

Measuring chaos



See also: https://www.physics.rutgers.edu/~haule/488/LogisticMap/Logistic%20map.html

Measuring chaos

According to the previous definition, $|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}$
the Lyapunov parameter λ is given by:

$$\lambda = \frac{1}{n}\ln\left|\frac{\Delta x_n}{\Delta x_0}\right| = \frac{1}{n}\ln\left|\frac{\Delta x_n}{\Delta x_{n-1}}\cdot\frac{\Delta x_{n-1}}{\Delta x_{n-2}}\cdot\frac{\Delta x_{n-2}}{\Delta x_{n-3}}\cdots\frac{\Delta x_1}{\Delta x_0}\right| = \frac{1}{n}\sum_{i=0}^{n-1}\ln\left|\frac{\Delta x_{i+1}}{\Delta x_i}\right|$$

If we consider the map as a function, we have:

$$x_{i+1} = f(x_i) \;\Rightarrow\; \Delta x_{i+1} = \Delta f(x_i) \;\Rightarrow\; \frac{\Delta x_{i+1}}{\Delta x_i} = \frac{\Delta f(x_i)}{\Delta x_i} = f'(x_i)$$

if the Δx_i are sufficiently small, which is true in case of convergence towards fixed points

hence:

$$\lambda = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right)$$

Measuring chaos

$$\lambda = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right)$$

for the logistic map:

$$f'(x_i) = 4r(1-2x_i)$$

a=4r	Lyap analitico	Lyap numerico
2.50	-0.693147	-0.694788
3.00	-0.000380	-0.002873
3.50	-0.872507	-0.873718
3.80	0.435603	0.434497
3.90	0.495620	0.494590
4.00	1.386294	1.384463

by Andrea Cacciola

See also: https://www.physics.rutgers.edu/~haule/488/LogisticMap/Logistic%20map.html

Measuring chaos

For the so-called *tent map*

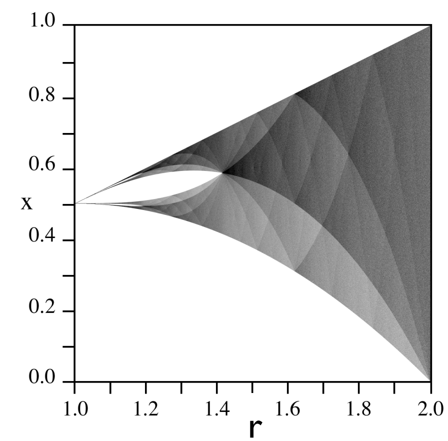
$$f(x) = \begin{cases} rx & 0 \leq x \leq 1/2 \\ r - rx & 1/2 \leq x \leq 1 \end{cases}$$

(for $0 \leq r \leq 2$ and $0 \leq x \leq 1$)

Since $f'(x) = \pm r$ for all x, we find

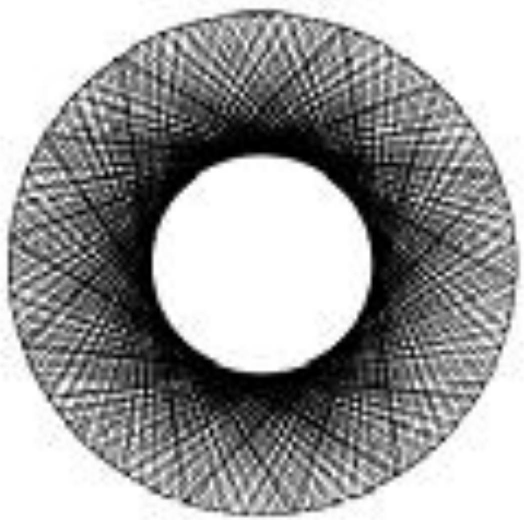
$$\begin{aligned} \lambda &= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right) \\ &= \lim_{n\to\infty} \left(\frac{\ln r}{n} \sum_{i=0}^{n-1} 1 \right) \\ &= \ln r \end{aligned}$$

This suggests that the tent map has chaotic solutions for all $r > 1$, since $\lambda = \ln r > 0$.



Billiards

NON Ergodicity
of circular
billiards



Ergodicity
of chaotic
billiards