ES 2

1) $\mathcal{A} = m \begin{pmatrix} 1 & \cos(x/\ell) \\ \cos(x/\ell) & 1 \end{pmatrix}$

$$\det \mathcal{A} = m^2 \left(1 - \cos^2(x/\ell)\right) \leftarrow = 0 \text{ per } \frac{x}{\ell} = \pm\frac{\pi}{2} + 2k\pi$$

$\mathcal{A}$ dev'essere invertibile, quindi i pti pti è mal def.

2) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(m\dot{x} + m\cos\left(\frac{x}{\ell}\right)\dot{y}\right) = m\ddot{x} - m\,\mathrm{sen}\left(\frac{x}{\ell}\right)\frac{\dot{x}\dot{y}}{\ell} + m\cos\left(\frac{x}{\ell}\right)\ddot{y}$$

$$\frac{\partial L}{\partial x} = -m\,\mathrm{sen}\left(\frac{x}{\ell}\right)\frac{\dot{x}\dot{y}}{\ell} - 4\alpha x(x^2 - a^2) - 8\mu\alpha x y^2$$

$$\ddot{x} + \cos\left(\frac{x}{\ell}\right)\ddot{y} + \frac{4\alpha}{m}x(x^2-a^2) + 8\frac{\mu\alpha}{m}xy^2 = 0 .$$

3)
- $x \to -x \quad \dot{x} \to -\dot{x}$
  $y \to -y \quad \dot{y} \to -\dot{y}$ $\longrightarrow$ no cost. del moto
- Inv. sotto trasl. temp. $\rightarrow$ Energia cost. del moto.

4) $$V = \alpha(x^2 - a^2)^2 + 4\alpha a^2(1-\mu)y^2 + 4\mu\alpha x^2 y^2$$

$$\frac{\partial V}{\partial x} = 4\alpha x(x^2 - a^2) + 8\mu\alpha x y^2 = 4\alpha x\left(x^2 + 2\mu y^2 - a^2\right)$$

$$\frac{\partial V}{\partial y} = 8\alpha a^2(1-\mu)y + 8\mu\alpha x^2 y = 8\alpha y\left(\mu x^2 - a^2(\mu - 1)\right)$$

Soluz. $\bar{\nabla} V = 0$:

$$y = 0 \quad e \quad 4\alpha x(x^2 - a^2) = 0 \rightarrow x = 0, a, -a)$$

$$x^2 = a^2 \frac{\mu-1}{\mu} \quad e \quad \#(\cancel{a^2} - \frac{a^2}{\mu} - \cancel{a^2} + 2\mu y^2) = 0 \rightarrow y = \pm \frac{a}{\mu\sqrt{2}}$$

$\uparrow$

$\exists$ se $\mu \geqslant 1$ $\qquad$ $\left(y^2 = \frac{a^2}{2\mu^2}\right)$

$\rightarrow$ Sempre sol: $(x,y) = (0,0), \ (0,-a), \ (0,a)$

Se $\mu \geqslant 1$: $(x,y) = \left(\pm a\sqrt{\frac{\mu-1}{\mu}}, \ \pm \frac{a}{\mu\sqrt{2}}\right)$ (4 soluz.)

$$\partial^2 V = \begin{pmatrix} 12\alpha x^2 - 4\alpha a^2 + 8\mu\alpha y^2 & 16\mu\alpha xy \\ 16\mu\alpha xy & 8\alpha\left(a^2(1-\mu) + \mu x^2\right) \end{pmatrix}$$

$$\partial^2 V(0,0) = \begin{pmatrix} -4\alpha a^2 & 0 \\ 0 & 8\alpha a^2(1-\mu) \end{pmatrix} \rightarrow \text{INSTAB.} \quad (\text{elem. } 11 < 0)$$

$$\partial^2 V(\pm a, 0) = \begin{pmatrix} 8\alpha a^2 & 0 \\ 0 & 8\alpha a^2 \end{pmatrix} \rightarrow \text{STAB.} \quad (\propto \mathbb{1}, \text{ autov pos})$$

$$\partial^2 V\left(\pm a\sqrt{\frac{\mu-1}{\mu}}, \pm \frac{a}{\mu\sqrt{2}}\right) = \begin{pmatrix} 8\alpha a^2\left(1-\frac{1}{\mu}\right) & 8\sqrt{2}\alpha a^2\sqrt{1-\frac{1}{\mu}} \\ 8\sqrt{2}\alpha a^2\sqrt{1-\frac{1}{\mu}} & 0 \end{pmatrix}$$

$\Rightarrow \det < 0 \Rightarrow$ INSTAB.

5) Pti stab. $(\pm a, 0)$

$$A = \mathcal{A}(\pm a, 0) = m \begin{pmatrix} 1 & \cos(a/\ell) \\ \cos(a/\ell) & 1 \end{pmatrix}$$

$$B = \partial^2 V(\pm a, 0) = 8\alpha a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(B - \lambda A) = \det \begin{pmatrix} 8\alpha a^2 - m\lambda & -m\lambda\cos(a/\ell) \\ -m\lambda\cos(a/\ell) & 8\alpha a^2 - m\lambda \end{pmatrix}$$

$$= m^2 \left[\left(8\frac{\alpha a^2}{m} - \lambda \right)^2 - \left(\cos\left(\frac{a}{\ell}\right) \lambda \right)^2 \right] =$$

$$= m^2 \left(8\frac{\alpha a^2}{m} - \lambda \left(1 + \cos\left(\frac{a}{\ell}\right)\right)\right)\left(\frac{8\alpha a^2}{m} - \lambda\left(1 - \cos\left(\frac{a}{\ell}\right)\right)\right)$$

$$= 0 \Rightarrow \quad \lambda_{\pm} = \frac{8\alpha a^2}{m\left(1 \pm \cos\left(\frac{a}{\ell}\right)\right)}$$

6) $\delta x = x - (\pm a) \qquad \delta y = y - 0$

$$L_{lin} = \frac{1}{2}(\delta\dot{x}, \delta\dot{y})\, A \begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} - \frac{1}{2}(\delta x, \delta y)\, B \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

7) $B - \lambda_+ A = \begin{pmatrix} 8\alpha a^2 - m\lambda_+ & -m\lambda_+ \cos(a/\ell) \\ -m\lambda_+ \cos(a/\ell) & 8\alpha a^2 - m\lambda_+ \end{pmatrix} =$

$$= \begin{pmatrix} 8\alpha a^2 - \frac{8\alpha a^2}{1+\cos(\frac{a}{\ell})} & -8\alpha a^2 \frac{\cos(a/\ell)}{1+\cos(a/\ell)} \\ -8\alpha a^2 \frac{\cos(a/\ell)}{1+\cos(a/\ell)} & 8\alpha a^2 - \frac{8\alpha a^2}{1+\cos(a/\ell)} \end{pmatrix} =$$

$$= 8\alpha a^2 \frac{\cos(\frac{a}{\ell})}{1+\cos(\frac{a}{\ell})} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$\Rightarrow$ Autovett. : $u_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B - \lambda_- A = \begin{pmatrix} 8\alpha a^2 - m\lambda_- & -m\lambda_- \cos(a/\ell) \\ -m\lambda_- \cos(a/\ell) & 8\alpha a^2 - m\lambda_- \end{pmatrix} =$

$$= \begin{pmatrix} 8\alpha a^2 - \frac{8\alpha a^2}{1-\cos(\frac{a}{\ell})} & -8\alpha a^2 \frac{\cos(a/\ell)}{1-\cos(a/\ell)} \\ -8\alpha a^2 \frac{\cos(a/\ell)}{1-\cos(a/\ell)} & 8\alpha a^2 - \frac{8\alpha a^2}{1-\cos(a/\ell)} \end{pmatrix} =$$

$$= 8\alpha a^2 \frac{\cos(\frac{a}{\ell})}{1+\cos(\frac{a}{\ell})} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

$\Rightarrow$ Autovett. $u_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Soluz. gen.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \pm a \\ 0 \end{pmatrix} + A_+ u_+ \cos(\sqrt{\lambda_+}\, t + \varphi_+) + $$
$$+ A_- u_- \cos(\sqrt{\lambda_-}\, t + \varphi_-)$$

8) Se $\mu = 0$ la Lagrangiana disaccoppia, cioè

$$L = \frac{m}{2}\dot{x}^2 - \alpha(x^2-a^2)^2 + \frac{m}{2}\dot{y}^2 - 4\alpha a^2 y^2$$
$$= L_x + L_y$$

Sia En. di L_x che En di L_y sono cost. del moto e siccome sono disaccoppiate in form. Ham. dovremo due variab. dinamiche che commutano, prod. in involuz. Inoltre commutano con Ham. di partenza $\Rightarrow$ $n=2$ cost. del moto in invol. in sist. a $n=2$ gradi di lib $\Rightarrow$ sist. integr.

$$H_x = \frac{p_x^2}{2m} + \alpha(x^2-a^2)^2$$

$$H_y = \frac{p_y^2}{2m} + 4\alpha a^2 y^2$$

$\{H_x, H_y\} = 0$ poiché $\{p_x, y\} = 0 = \{p_y, x\}$ etc.

Inoltre $\{H_x, \underbrace{H_x + H_y}_{H}\} = \underbrace{\{H_x, H_x\}}_{=\,0 \text{ su}} + \underbrace{\{H_x, H_y\}}_{=\,0} = 0$

ovunque

p.p.

analogo su $\{H_y, H\} = 0.$

ES. 1 | 4) $L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - V(\sqrt{x^2+y^2}) \to p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

L invariante sotto

$$\bar{\varphi}(\bar{r},\alpha) = \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\bar{\psi}(\bar{r},\dot{\bar{r}},\alpha) = \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$\to$ Cost. del moto

$$P = \sum_{h=1}^{2} \frac{\partial \varphi_h}{\partial \alpha}(\bar{r},0)\, p_h = \frac{\partial \bar{\varphi}}{\partial \alpha}(\bar{r},0) \cdot \bar{p} =$$

$$= (-y\, p_x + x\, p_y) .$$

5) $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$p_r = m\dot{r} \qquad p_\theta = mr^2\dot{\theta} \quad \to \quad \dot{r} = \frac{p_r}{m} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$

$$H = \left(p_r\dot{r} + p_\theta\dot{\theta} - L\right)\Big|_{\dot{r}=\frac{p_r}{m} \;\; \dot{\theta}=\frac{p_\theta}{mr^2}} =$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$ $\to \theta$ coord. ciclica

$\frac{\partial H}{\partial q_1} = 0 \Rightarrow \dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0 \Rightarrow p_1$ è cost. del moto

$\Rightarrow$ θ ciclica implica che p_θ è cost. del moto.

p_θ genera rotaz. sul piano

$$x p_y - y p_x = m(x\dot{y} - y\dot{x}) =$$

$$= m\left(r\cos\theta \left(\cancel{\dot{r}\,\mathrm{sen}\theta} + r\dot{\theta}\cos\theta\right) - r\,\mathrm{sen}\theta\left(\cancel{\dot{r}\cos\theta} - r\dot{\theta}\,\mathrm{sen}\theta\right)\right)$$

$$= 2mr\dot{\theta} = 2p_\theta .$$

$$\left\{ p_\theta, \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m} + V(r) \right\} = 0$$

perché $\{p_\theta, p_\theta\} = 0 \quad \{p_\theta, p_r\} = 0 \quad \{p_\theta, r\} = 0$

da P.P. fondamentali.