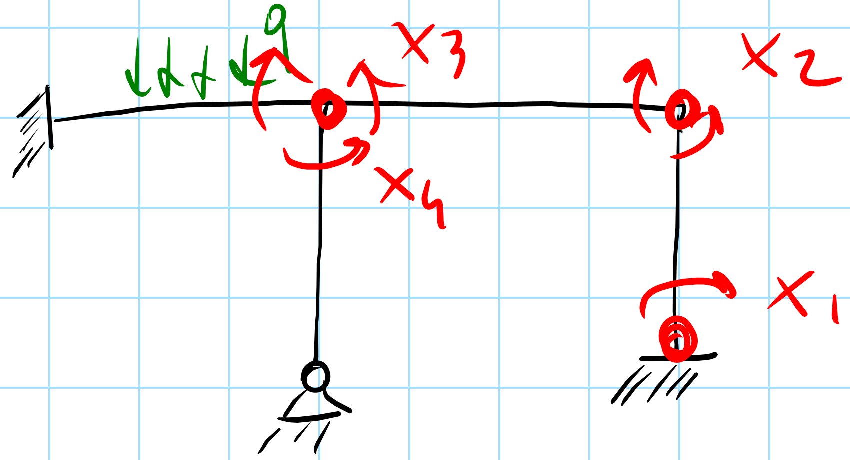
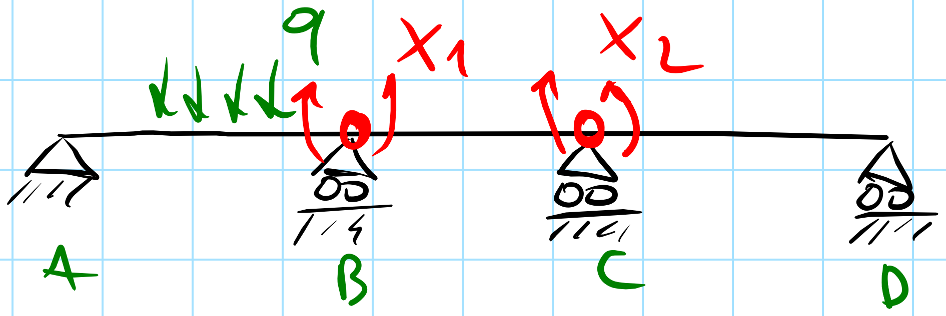
ELASTIC STRUCTURES: METHODS OF SOLUTION (REVIEW)





(HYPERSTATICS)
(REDUNDANT)

φ : ROTATION

→ 1° METHOD: "METHOD OF FORCES" (HYPERSTATIC REACTIONS AS UNKNOWNS)

SOLUTION VIA COMPATIBILITY EQS $[\varphi_{BA} = \varphi_{BC} \; ; \; \varphi_{CB} = \varphi_{CD}]$ 2 EQS FOR TWO UNKNOWNS (X_1, X_2)

⇒ LINEAR SYSTEM: $[\underset{\sim}{C}][\underline{X}] = [\underline{U}]$

CEDIBILITY MATRIX (FLEXIBILITY MATRIX)

UNKNOWNS

KNOWN TERMS

$$[\underline{X}] = [\underset{\sim}{C}^{-1}][\underline{U}]$$

– 2° METHOD : DISPLACEMENT METHOD

↳ UKNOWNS

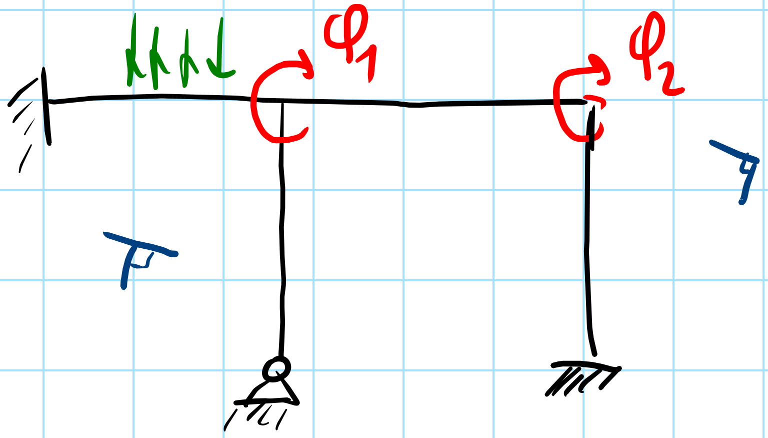
$$[\underset{\sim}{K}]\,[\underline{u}] = [\underline{F}]$$

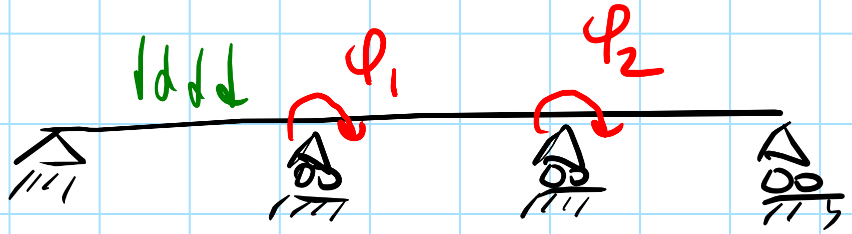
NODAL FORCES

VECTOR OF UNKNOWNS $\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$

STIFFNESS MATRIX

SOLUTION BY EQUILIBRIUM EQS

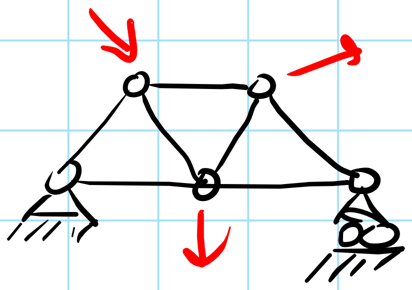




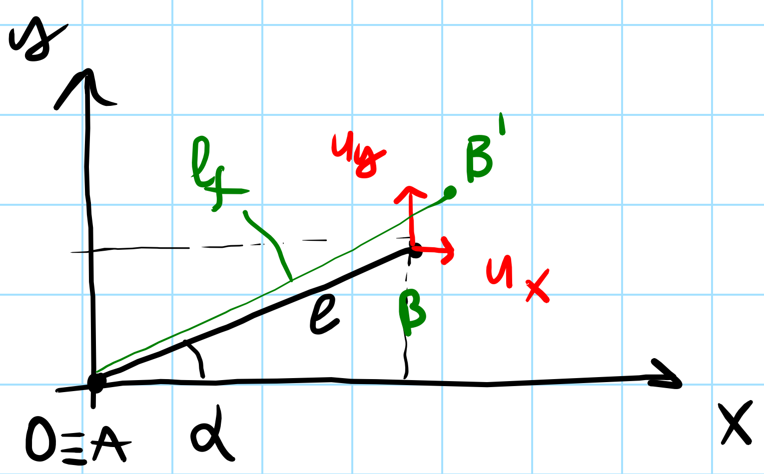
FEM formulated starting from STATIONARY TOTAL POTENTIAL ENERGY

$\Longrightarrow$ LINEAR SYSTEM : $\underset{\sim}{K}\,\underline{u} = \underline{F}$

DISPLACEMENT METHOD FOR PLANE TRUSS STRUCTURES



KINEMATICS OF A BAR IN THE PLANE



$$l = \sqrt{x_B^2 + y_B^2} \quad ; \quad l_f = \sqrt{(x_B+u_x)^2 + (y_B+u_y)^2} = l_f(u_x, u_y)$$

$$\boxed{u_x, u_y \ll l}$$

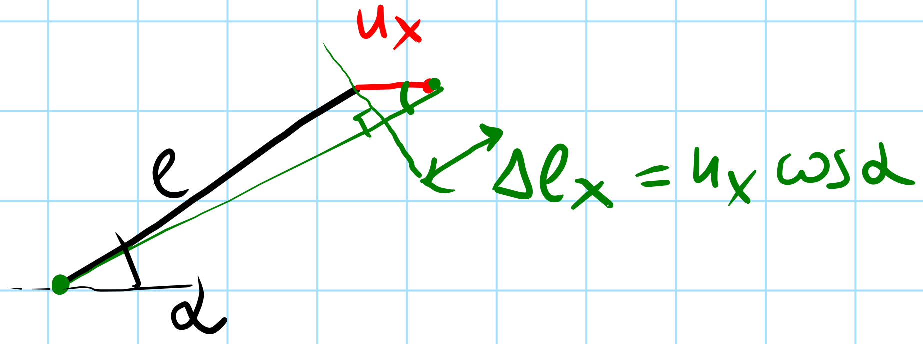
GOAL: LINEARIZE $l_f(u_x, u_y)$

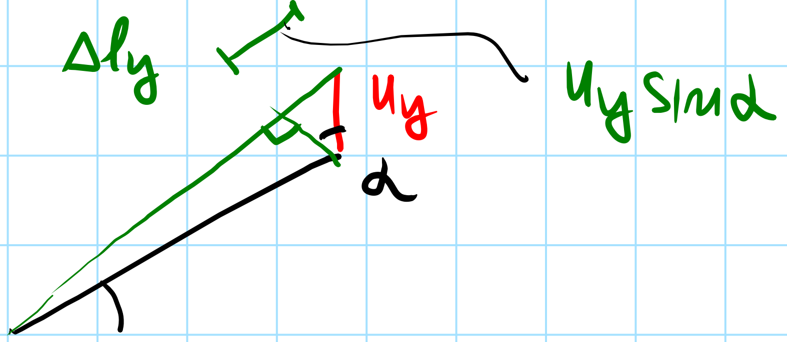
$$l_f = l_f(0,0) + \left.\frac{\partial l_f}{\partial u_x}\right|_{(0,0)} u_x + \left.\frac{\partial l_f}{\partial u_y}\right|_{(0,0)} u_y + O\left(\sqrt{u_x^2+u_y^2}\right)$$

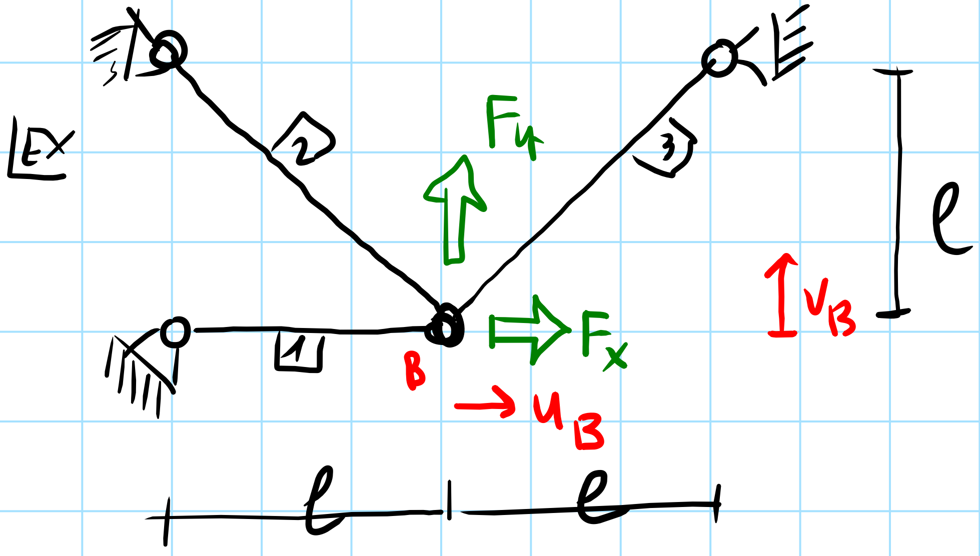
$$l_f \simeq l + \left[\frac{1}{\cancel{2}\sqrt{\ldots}} \cancel{2}(x_B+\cancel{u_x})\right]_{(0,0)} u_x + \left[\frac{1}{\cancel{2}\sqrt{\ldots}} \cancel{2}(y_B+\cancel{u_y})\right]_{(0,0)} u_y$$

$$l_f \simeq l + \underbrace{\frac{1}{l} x_B}_{\cos\alpha} u_x + \underbrace{\frac{1}{l} y_B}_{\sin\alpha} u_y \longrightarrow \underbrace{l_f - l}_{\Delta l} = \underbrace{\cos\alpha\, u_x}_{\Delta l_x} + \underbrace{\sin\alpha\, u_y}_{\Delta l_y}$$

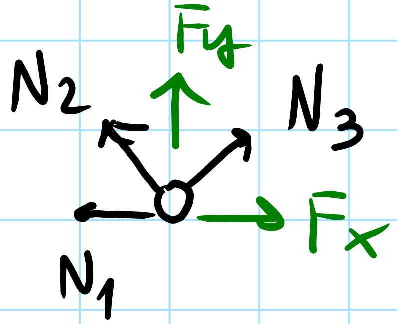
CHANGE IN LENGTH OF A BAR IS THE SUM OF TWO CONTRIBUTIONS







SOLVE BY DISPL. METHOD (EQUILIBRIUM EQS OF THE NODE B!)



$$\stackrel{+}{\rightarrow}: \ -N_1 - N_2 \frac{1}{\sqrt{2}} + N_3 \frac{1}{\sqrt{2}} + F_x = 0$$

$$+\uparrow: \ +N_2 \frac{1}{\sqrt{2}} + N_3 \frac{1}{\sqrt{2}} + F_y = 0$$

2 D.O.Fs : u_B, v_B | LOADS: F_x, F_y

$\ell_1 = \ell \qquad A_1 = A$

$\ell_2 = \ell\sqrt{2} \qquad A_2 = \sqrt{2}A$

$\ell_3 = \ell\sqrt{3} \qquad A_3 = \sqrt{2}A$

E : cost

USE CONSTITUTIVE LAW : $\quad N = \frac{EA}{\ell}\Delta\ell$

$N_1 = \frac{EA}{\ell}\Delta\ell_1$

$N_2 = \frac{E\sqrt{2}A}{\ell\sqrt{2}}\Delta\ell_2$

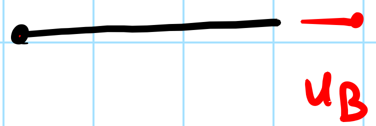
$N_3 = \frac{E\sqrt{2}A}{\ell\sqrt{2}}\Delta\ell_3$

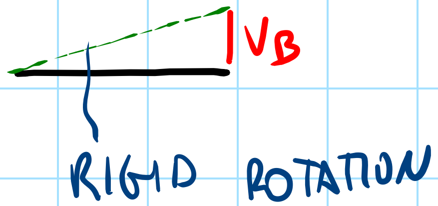
THE KEY IS NOW TO EXPRESS

$$\Delta\ell_i = \Delta\ell_i(u_B, v_B)$$

u_B | v_B | Δl_i

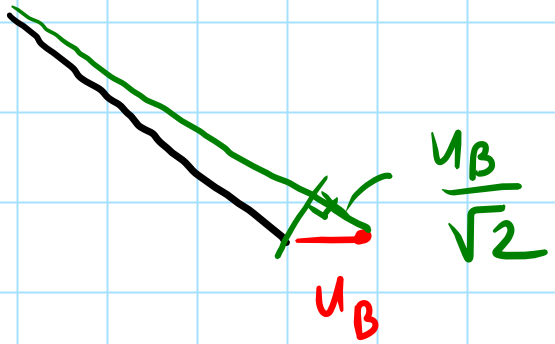
1)

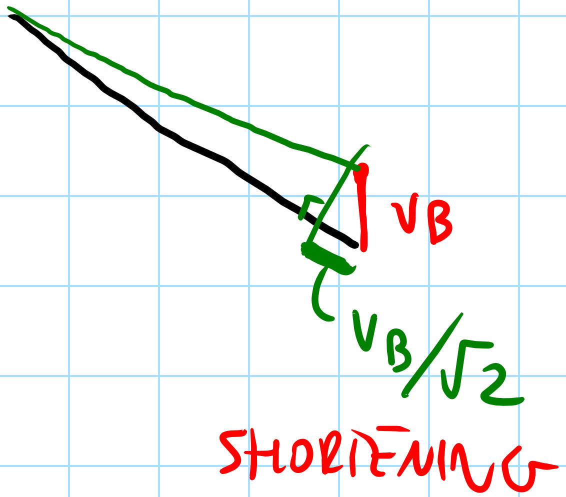




$$\boxed{\Delta l_1 = +u_B} + 0$$

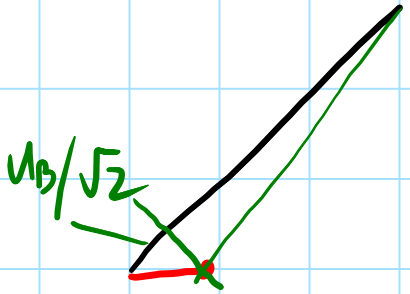
2)

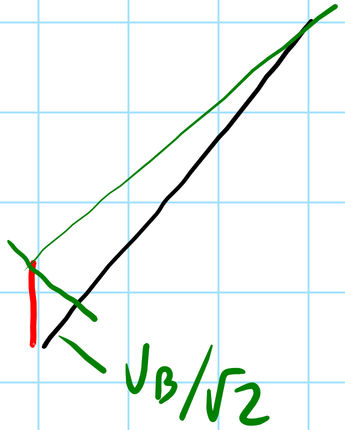




$$\Delta l_2 = +\frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$$

3)





$$\Delta l_3 = -\frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$$

$$\frac{EA_i}{l_i} = K$$

EQUIL EQS :
$$\left.\begin{aligned}&-K u_B - \frac{1}{\sqrt{2}} K \frac{1}{\sqrt{2}}(u_B - v_B) + \frac{1}{\sqrt{2}} K \frac{1}{\sqrt{2}}(-u_B - v_B) + F_x = 0\\ &+\frac{1}{\sqrt{2}} K \frac{1}{\sqrt{2}}(u_B - v_B) + \frac{1}{\sqrt{2}} K \frac{1}{\sqrt{2}}(-u_B - v_B) + F_y = 0\end{aligned}\right\}$$

$$-[K][u]+[F]=0$$

$$-\underbrace{\begin{bmatrix} 2K & 0 \\ 0 & K \end{bmatrix}}_{\underset{\sim}{K}\ (2\times2)} \underbrace{\begin{bmatrix} u_B \\ v_B \end{bmatrix}}_{\underline{u}} + \underbrace{\begin{bmatrix} F_x \\ F_y \end{bmatrix}}_{\underline{F}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \boxed{\underset{\sim}{K}\,\underline{u} = \underline{F}}$$

KNOWN

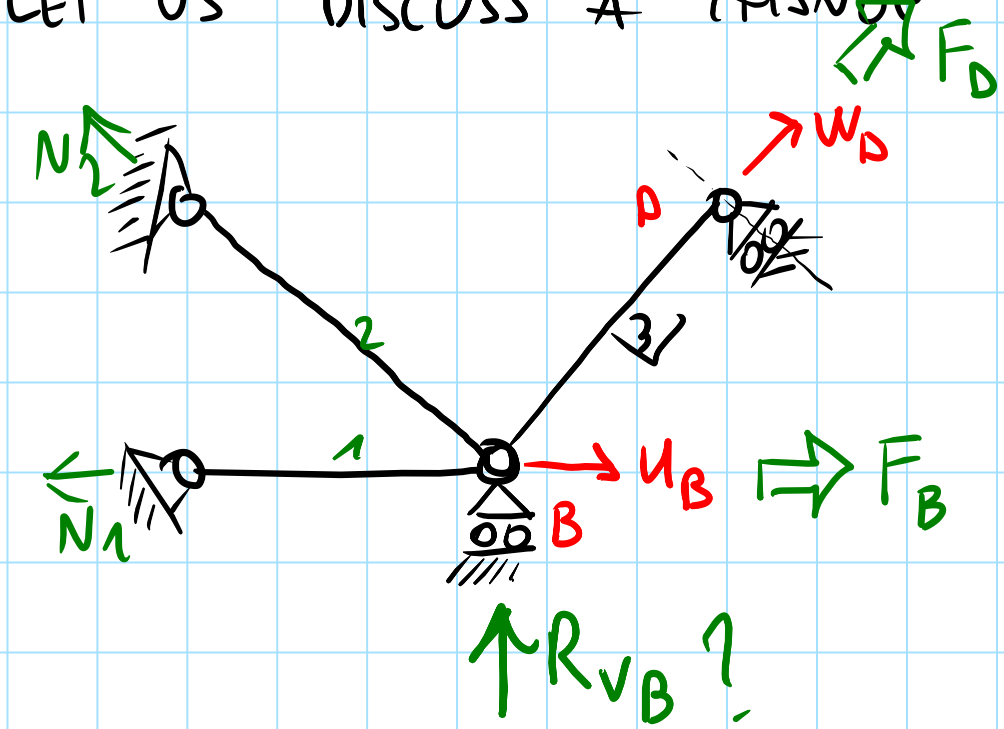
UNKNOWNS (u_B, v_B)

STIFFNESS MATRIX

$\underset{\sim}{K}$: STIFFNESS MATRIX ($m \times m$; $m = n°$ OF UNKNOWNS)

$\underset{\sim}{K}$: POSITIVE DEFINITE
$\underset{\sim}{K}$: SYMMETRIC
} PROPERTIES IN LINEAR ELASTICITY ($\underset{\sim}{K}^{-1}$ ALWAYS EXIST !) $\underline{u} = \underset{\sim}{K}^{-1}\,\underline{F}$

LET US DISCUSS A CHANGE IN THE EXERCISE



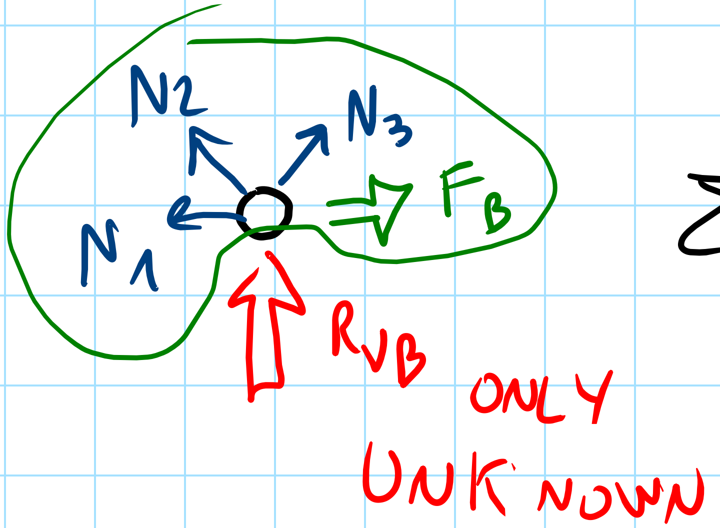
DOFs ? 2 : u_B, w_D $\Rightarrow$ EQUIL. EQS

$$\sum F_{u_B} = 0$$

$$\sum F_{w_D} = 0$$

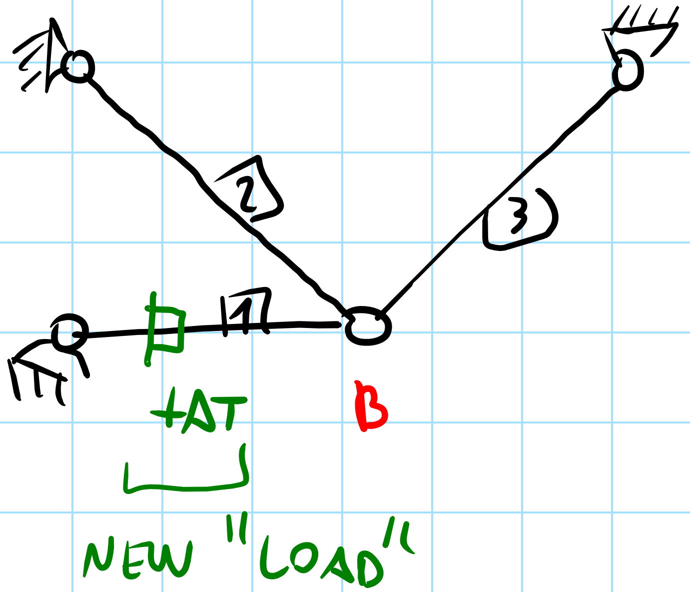
IN ORDER TO FIND R_{V_B} WE IMPOSE, AFTER HAVING SOLVED THE PROBLEM

$\underset{\sim}{K}\,\underline{u} = \underline{F}$, VERTICAL EQUILIBRIUM ON NODE B $\left(+\!\uparrow \sum F_{B_y} = 0\right)$.



$$\sum F_{B_y} = 0 \Rightarrow R_{V_B}!$$

EFFECTS OF A THERMAL VARIATION



EQUIL OF B:

$\xrightarrow{+}$: $-N_1 - N_2 \frac{1}{\sqrt{2}} + N_3 \frac{1}{\sqrt{2}} = 0$

$+\uparrow$: $+N_2 \frac{1}{\sqrt{2}} + N_3 \frac{1}{\sqrt{2}} = 0$

NO APPLIED FORCES! (NO F_x, F_y!)

$$N_1 = \frac{EA}{\ell} \Delta \ell^{el} = \frac{EA}{\ell} \left(\Delta \ell - \alpha \Delta T \ell \right) = \frac{EA}{\ell} \left(u_B - \alpha \Delta T \ell \right)$$

OTHER N_2, N_3 ARE UNCHANGED

$$\underset{\underset{\sim}{K}}{\begin{bmatrix} 2K & 0 \\ 0 & K \end{bmatrix}} \underset{\underline{u}}{\begin{bmatrix} u_B \\ v_B \end{bmatrix}} = \underset{\underline{F}}{\begin{bmatrix} K \alpha \Delta T \ell \\ 0 \end{bmatrix}}$$

$\underline{u}$ IS A FUNCTION OF THE "THERMAL LOAD" ΔT

$$\underset{\sim}{K} \, \underline{u} = \underline{F}^{el} + \underline{F}^{inel} + \underline{F}^{DISP.*}$$

*: DISP.: DISPLACEMENT APPLIED TO CONSTRAINTS