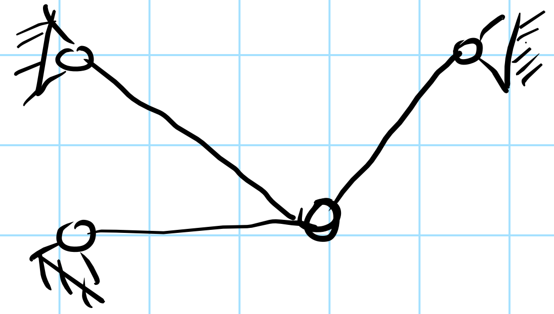
DISPLACEMENT METHOD : $\underset{\sim}{K}\,\underline{u} = \underline{F}$ (*)

CSM, 24/02/26

STIFFNESS MATRIX $(m \times m)$; $m = n°$ OF UNKNOWNS

$\underset{\sim}{K}$ depends on geometry, mechanical properties, constraints



$\Downarrow$

$\underset{\sim}{K}$

$\underset{\sim}{K}$ IS 1) SYMMETRIC
2) POSITIVE DEFINITE.

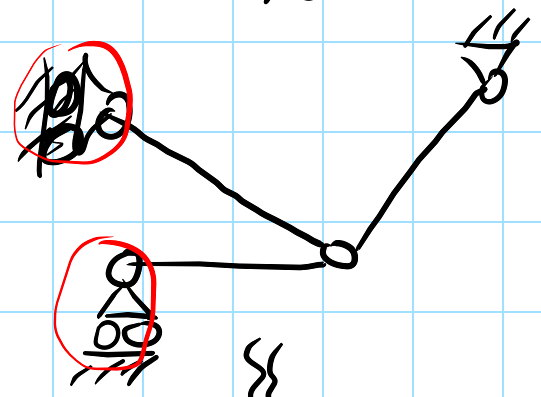
2) POSITIVE DEFINITE?

QUADRATIC FORM

$\underset{\sim}{K}\,\underline{u} \cdot \underline{u} > 0$ FOR $\underline{u} \neq \underline{0}$

$\underset{\sim}{K}\,\underline{u} \cdot \underline{u} = 0$ ONLY FOR $\underline{u} = \underline{0}$

DEFINITION OF POSITIVE DEFINITE



$\Downarrow$

DIFFERENT $\underset{\sim}{K}$

TH. OF CLAPEYRON

$\Phi = \frac{1}{2}\,\underline{F} \cdot \underline{u} > 0$ (STRAIN ENERGY) (*)

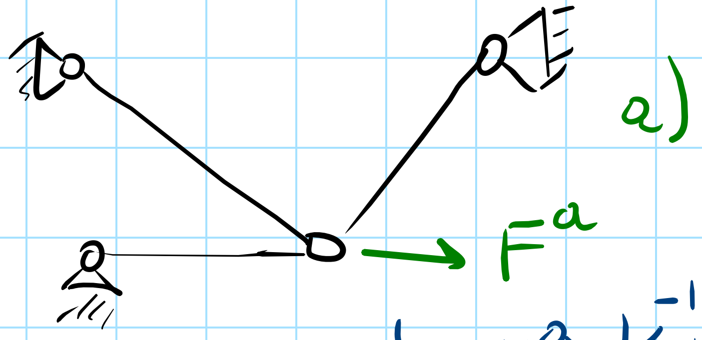
$\Phi = \frac{1}{2}\,\underset{\sim}{K}\,\underline{u} \cdot \underline{u} > 0$

$\underset{\sim}{K}\,\underline{u} \cdot \underline{u} > 0$, $\underline{u} \neq \underline{0}$

$\ldots = 0$, $\underline{u} = \underline{0}$

2) SYMMETRY $\left(\underset{\sim}{K} = \underset{\sim}{K}^T\right) \Rightarrow$ BECAUSE TH. OF BETTI:

$$\underline{F}^a \cdot \underline{u}^b = \underline{F}^b \cdot \underline{u}^a$$

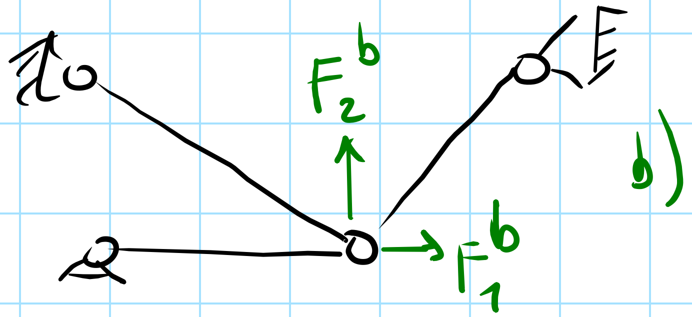


$\hookrightarrow \underline{u}^a = \underset{\sim}{K}^{-1} \underline{F}^a$

$$\underset{\sim}{K}\,\underline{u}^a \cdot \underline{u}^b = \underset{\sim}{K}\,\underline{u}^b \cdot \underline{u}^a$$

$$\underline{u}^a \cdot \underset{\sim}{K}^T \underline{u}^b = \underset{\sim}{K}\,\underline{u}^b \cdot \underline{u}^a$$

$\underset{\sim}{K}^T = \underset{\sim}{K}$



$\hookrightarrow \underline{u}^b = \underset{\sim}{K}^{-1} \underline{F}^b$

HOW TO COMPUTE ELEMENTS K_{ij} OF $\underset{\sim}{K}$?

- WRITE EQUILIBRIUM EQS. AND EXTRACT FROM THE LINEAR SYSTEM THE WANTED COEFF. K_{ij}

$$\underset{\sim}{K} = \begin{bmatrix} 2k & 0 \\ 0 & k \end{bmatrix}$$

EX: $K_{11} u_1 + K_{12} u_2 + K_{13} u_3 \;\cdots\cdots = F_1$

- ANOTHER APPROACH (K_{ij}?)

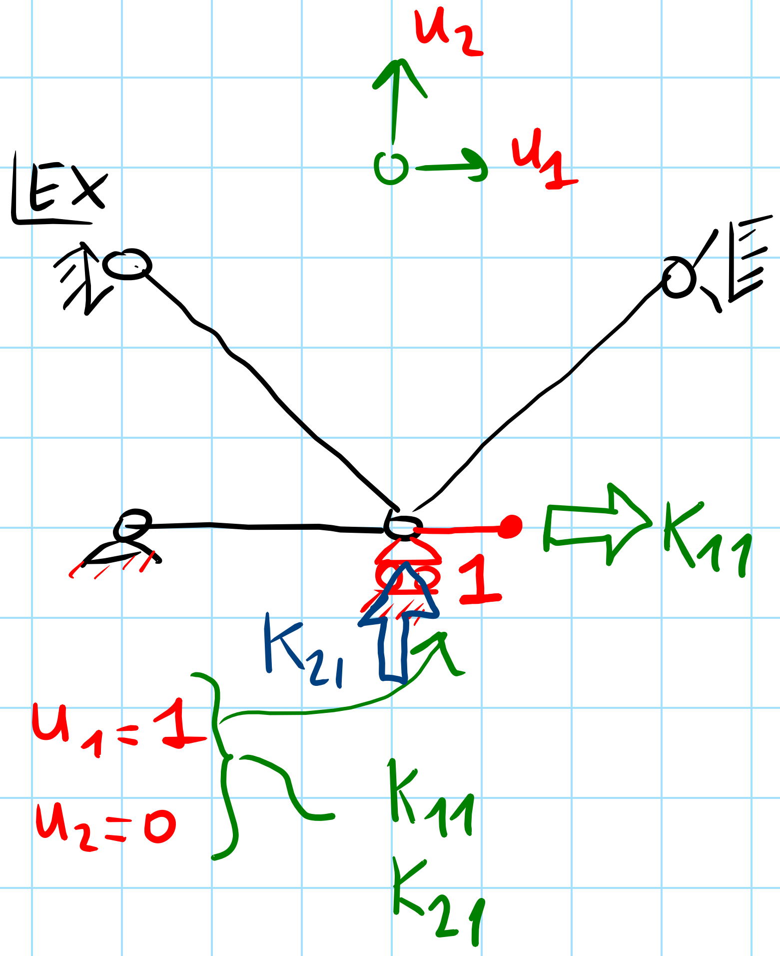
$$\left.\begin{aligned} K_{11} u_1 + K_{12} u_2 &= F_1 \\ K_{21} u_1 + K_{22} u_2 &= F_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &\boxed{u_1 = 1} \\ &u_2 = 0 \end{aligned} \Rightarrow \left.\begin{aligned} K_{11} &= F_1 \\ K_{21} &= F_2 \end{aligned}\right\}$$

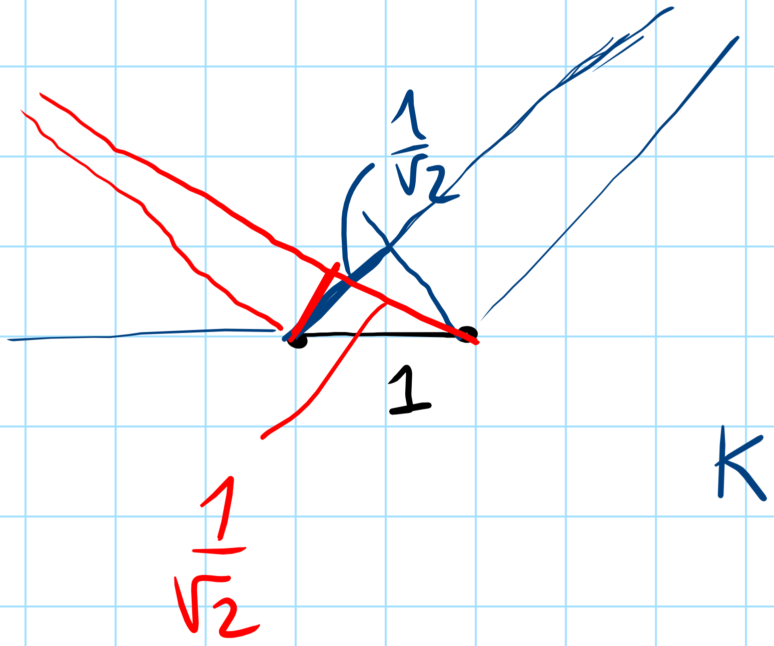
IF $u_i = 1$ ($u_j = 0$, $i \neq j$)

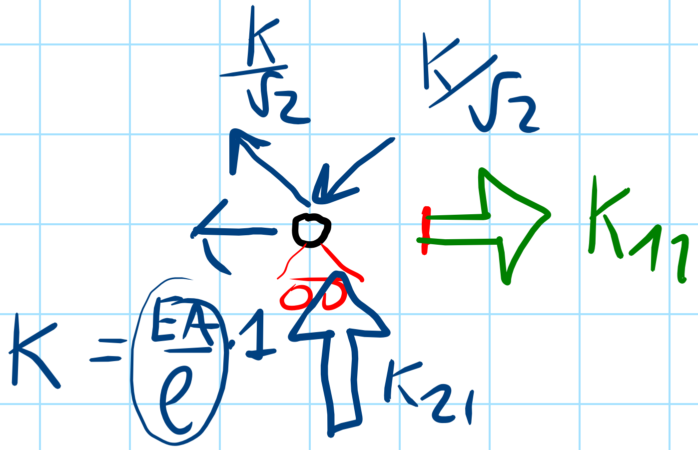
$K_{1i} = F_1$ $\quad$ $K_{3i} = F_3$

$K_{2i} = F_2$ $\quad$ $\vdots$

K_{ij}? IT IS THE "FORCE" ASSOCIATED WITH D.O.F. u_i WHEN $u_j = 1$ AND THE OTHER $(u_p,\ p \neq j)$ ARE ALL EQUAL TO 0.





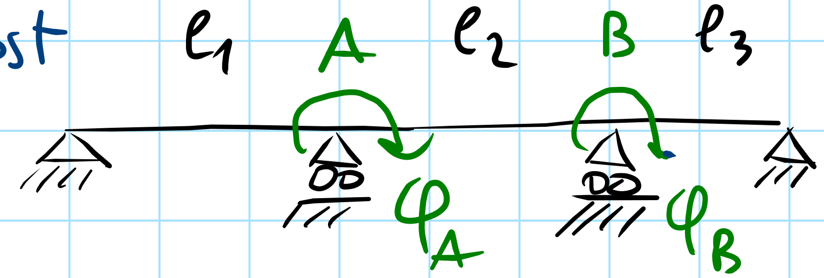


$\xrightarrow{+}$: $-K - \frac{K}{2} - \frac{K}{2} + K_{11} = 0$

$$\boxed{K_{11} = 2K}$$

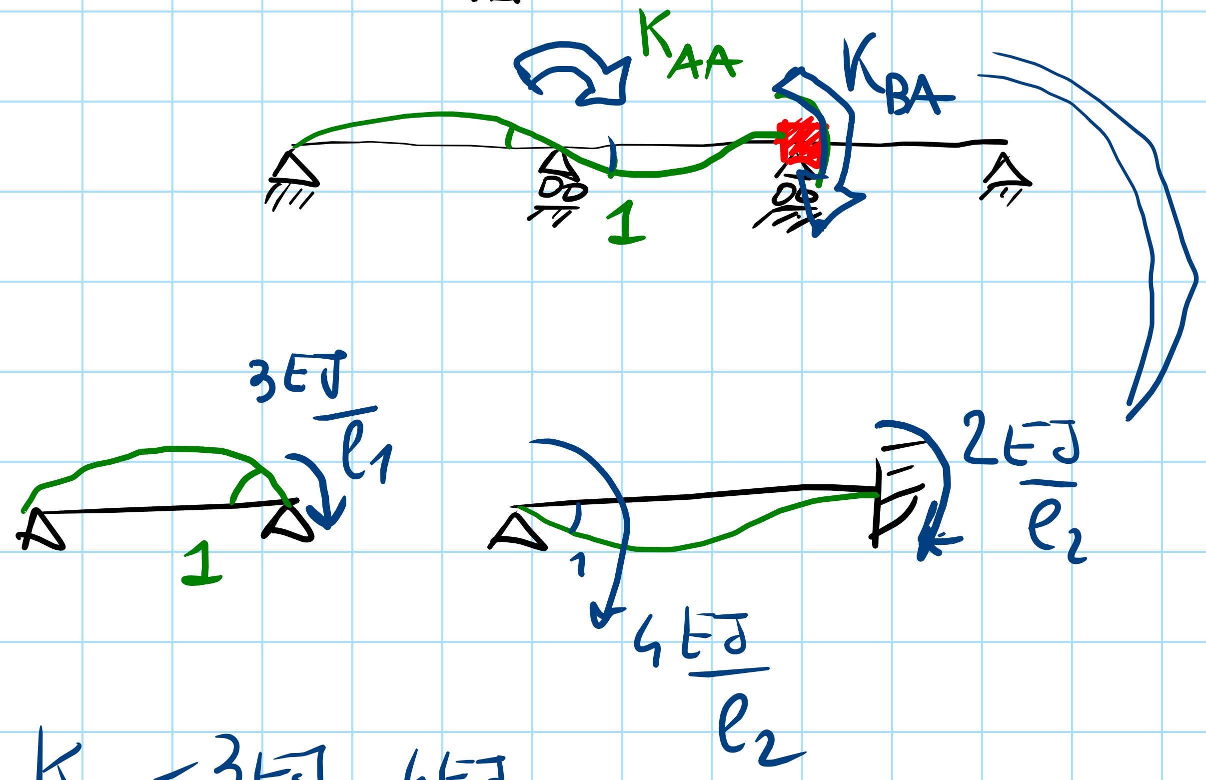
$+\uparrow$: $\cancel{\frac{K}{2}} - \cancel{\frac{K}{2}} + K_{21} = 0 \implies \boxed{K_{21} = 0}$

EX EJ cost



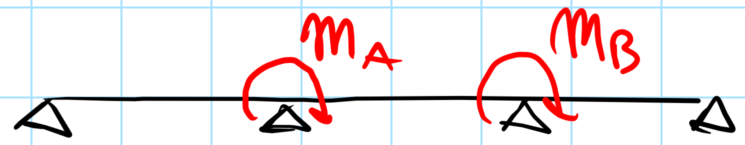
DOFS

$$\left.\begin{array}{l}\varphi_A = 1 \\ \varphi_B = 0\end{array}\right\} \Rightarrow \begin{array}{l} K_{AA} \\ K_{BA} \end{array}$$



$$K_{AA} = \frac{3EJ}{l_1} + \frac{4EJ}{l_2}$$

$$K_{BA} = 2\frac{EJ}{l_2}$$



$$\left[\ \underset{\sim}{K}\ \right]\begin{bmatrix}\varphi_A \\ \varphi_B\end{bmatrix} = \begin{bmatrix} m_A \\ m_B \end{bmatrix}$$

$$\underset{\sim}{K} = \begin{bmatrix} \frac{3EJ}{l_1} + \frac{4EJ}{l_2} & \bigcirc \\ \frac{2EJ}{l_2} & \cdots \end{bmatrix}$$

INTRODUCTION TO THE CALCULUS OF VARIATIONS

MANY ENGINEERING PROBLEMS CAN BE SOLVED LOOKING FOR EXTREMAL VALUES (MAXIMUM / MINIMUM) OF AN INTEGRAL-DIFFERENTIAL EXPRESSION

HYPOTHESES: 1D DOMAIN (x); $y=f(x)$ IS KNOWN, an integral-differential expression involving $y=f(x)$ e y', y'' can be written as:

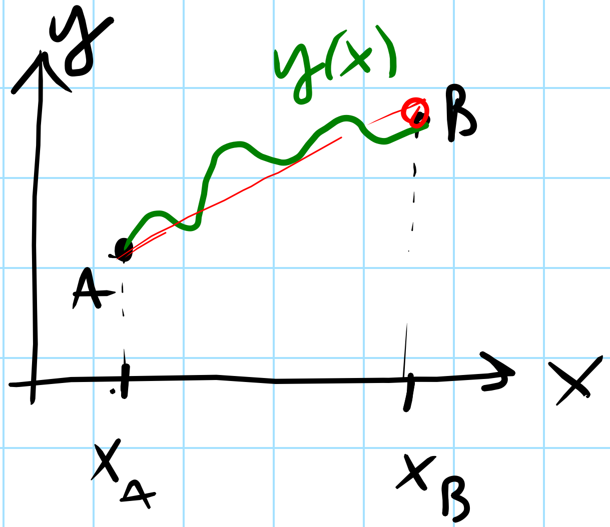
$$I(y) = \int_{x_A}^{x_B} F(x; y, y', y'', \dots)\, dx$$

defines the problem

$I(y)$: FUNCTIONAL = FUNCTION OF A FUNCTION

$$y(x) \longrightarrow I(y) \in \mathbb{R}$$

EX

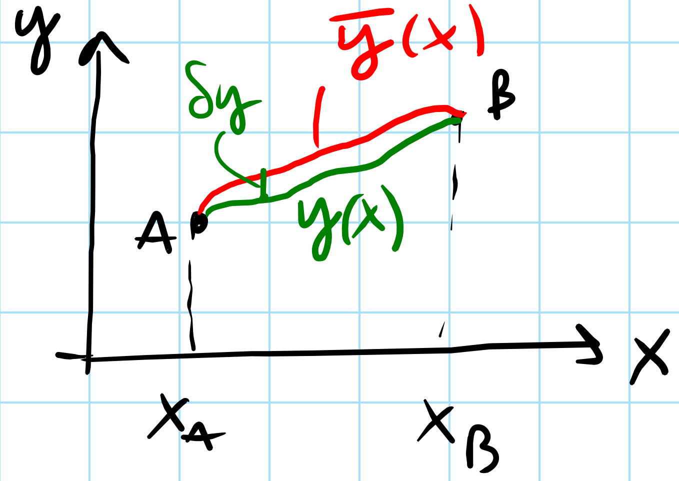


LENGTH OF THE CURVE

$$I(y) = \int_A^B \sqrt{dx^2 + dy^2} = \int_{x_A}^{x_B} \underbrace{\sqrt{1 + y'^2(x)}}_{F(x; y')}\, dx$$

PROBLEM: FIND $y(x)$ THAT MINIMIZES I AMONG ALL CONTINUOUS FUNCTIONS HAVING $y'(x)$ CONTINUOUS

CONCEPT OF VARIATION



$\bar{y}(x)$ IS CLOSE TO $y(x) \Rightarrow$ VARIATION $\boxed{\delta y} = \bar{y}(x) - y(x)$

NOTE THAT $\delta y(x_A) = \delta y(x_B) = 0$

PROPERTIES OF δy $\rightarrow$ $\frac{d}{dx}(\delta y) = \frac{d}{dx}(\bar{y} - y) = \bar{y}' - y' = \delta y'$

$\searrow$ $\int_{x_A}^{x_B} \delta y \, dx \Rightarrow \delta \int_{x_A}^{x_B} y \, dx$

GIVEN $I(y)$

$$\delta I = \delta \int_{x_A}^{x_B} F(x, y, y') \, dx = \int_{x_A}^{x_B} \delta F(x, y, y') \, dx \,, \quad \text{BUT} \quad \delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y'$$

FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS:

IF $\int_{x_A}^{x_B} y(x) \, \delta y \, dx = 0 \,, \ \forall \delta y \Rightarrow y(x) = 0$ IN $[x_A, x_B]$

STATIONARITY OF $I(y)$ WITH $\boxed{y(x_A) = y_A , \; y(x_B) = y_B}$ ($\delta y = 0$, $\delta y = 0$)

$\int f g' = [fg] - \int f' g$

The stationarity condition of $I(y)$ can be written as

$$\boxed{\delta I = 0 , \; \forall \delta y} \Longrightarrow y^*(x) \text{ SOLUTION}$$

$$\delta I = \int_{x_A}^{x_B} \delta F \, dx = \boxed{\int_{x_A}^{x_B} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx = 0 , \; \forall \delta y}$$

INTEGRATE BY PARTS : $\displaystyle \int_{x_A}^{x_B} \frac{\partial F}{\partial y'} \delta y' = \left[\frac{\partial F}{\partial y'} \delta y \right]_{x_A}^{x_B} - \int_{x_A}^{x_B} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y \, dx$

OUR STATEMENT BECOMES :

$$\left[\int_{x_A}^{x_B} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] \delta y \, dx + \underbrace{\left[\frac{\partial F}{\partial y'} \underset{0}{\delta y} \right]_{x_A}^{x_B}}_{0} = 0 , \; \forall \delta y \right]$$

FOR THE "LEMMA"

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0}$$

IN THE DOMAIN

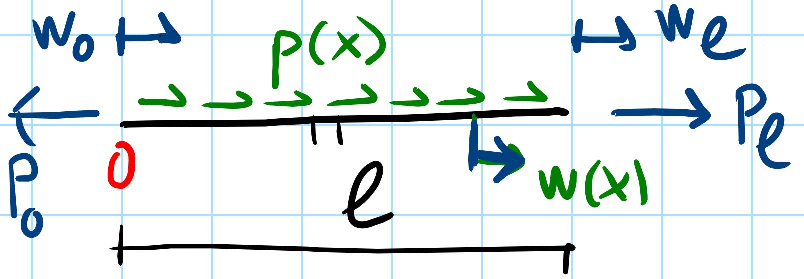
$x \in [x_A, x_B]$

$$\boxed{\delta I = 0, \ \forall \delta y} \iff \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{WITH} \ \left[\begin{array}{l} y(x_A) = y_A \\ y(x_B) = y_B \end{array}\right]}$$

$(\delta y = 0$ IN $x_A, x_B)$

EULER-LAGRANGE EQ OF THE PROBLEM.

EX: EUL-LAGR EQ FOR THE FUNCTIONAL TOTAL POTENTIAL ENERGY (T.P.E.) OF A BAR



$\varepsilon(x) = w'(x)$ AXIAL STRAIN

DENSITY OF STRAIN ENERGY $\varphi = \frac{1}{2} E \varepsilon^2$ (ENERGY PER UNIT VOLUME)

$$\Pi(w) = \underbrace{\int_0^\ell \frac{1}{2} E (w'(x))^2 \underbrace{A\,dx}_{dV}}_{\text{STRAIN ENERGY}} - \underbrace{\int_0^\ell p(x)\, w(x)\, dx - P_\ell w_\ell + P_0 w_0}_{\text{POTENTIAL OF LOADS}}$$

$$= \int_0^\ell \underbrace{\frac{1}{2} E A w'^2 - p w}_{F(x, w, w')} \, dx - \Big[P w \Big]_0^\ell$$

NOTE THAT, HERE, $w(0) = w_0$ AND $w(\ell) = w_\ell$ CAN VARY BECAUSE IN THE SKETCH THERE ARE NO CONSTRAINTS!

IN ORDER TO FIND THE GOVERNING EQ (I.E. EULER-LANGRANGE EQ OF THE PROBLEM) WE HAVE TWO OPTIONS.

1) $F = \frac{1}{2} EAw'^2 - pw \Longrightarrow \frac{\partial F}{\partial w} = -p \quad , \quad \frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right) = \frac{d}{dx}(EAw')$

EUL-LANGR. EQ: $\boxed{-p(x) - (EA(x)\, w'(x))' = 0}$ EQUIL. EQ OF A LOADED BAR.

$\int \underbrace{EAw'}_{f} \underbrace{\delta w'}_{g'} = [EAw'\delta w]_0^\ell - \int_0^\ell (EAw')'\delta w$

NOTE: EA CONST $\Longrightarrow$ EA $w''(x) = -p(x)$

2) $\boxed{\delta\Pi = 0, \delta w} \Longrightarrow w^*(x)$ SOLUTION

$$\delta\Pi = \int_0^\ell (\underbrace{EAw'\delta w'}_{\text{BY PARTS}} - p\,\delta w)\, dx - [P\delta w]_0^\ell$$

$$\boxed{\delta\Pi = -\int_0^\ell \underbrace{[(EAw')' + p]}_{=0} \delta w\, dx - \underbrace{[(P - EAw')\delta w]_0^\ell}_{0 \to \text{B.CONDITIONS (NEXT PAGE)}} = 0 \quad \forall \delta w}$$

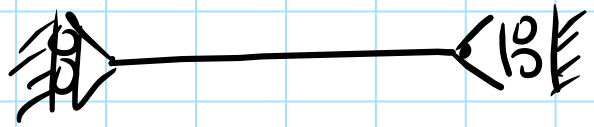
$(EAw')' + p = 0$ (E-L)

LET US DISCUSS BOUNDARY CONDITIONS:

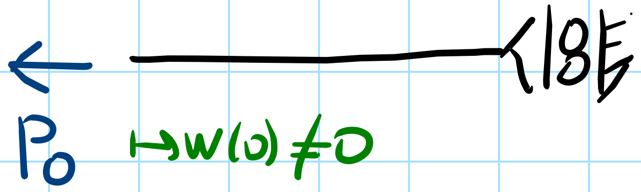
$$[P(\ell) - EA\,w'(\ell)]\,\delta w(\ell) - [P(0) - EA\,w'(0)]\,\delta w(0) = 0$$

[4 COMBINATIONS]

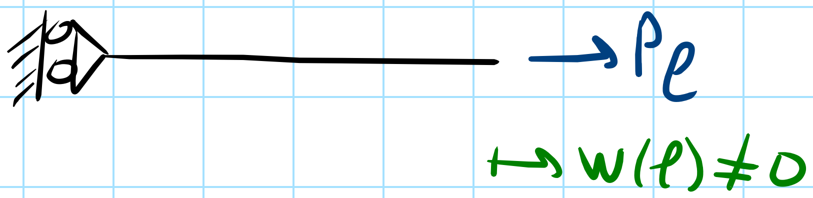
$\delta w(0) = 0 \,, \quad \delta w(\ell) = 0$



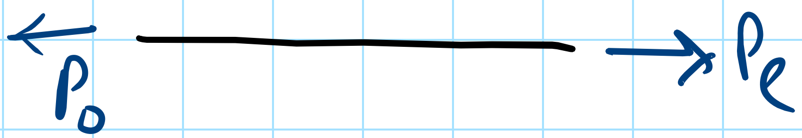
$P(0) = EA\,w'(0) \,, \quad \delta w(\ell) = 0$



$\delta w(0) = 0 \,, \quad P(\ell) = EA\,w'(\ell)$

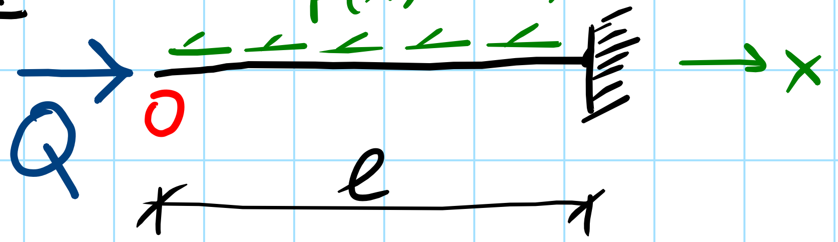


$P(0) = EA\,w'(0) \,, \quad P(\ell) = EA\,w'(\ell)$



EX

$p(x) = \alpha x \quad (\alpha \in \mathbb{R})$



EA: CONST

$$\begin{cases} EA\,w''(x) = -(-\alpha x) \\ -Q = EA\,w'(0) \\ w(\ell) = 0 \end{cases}$$

2° ORDER ODE

SET OF 2 B.CONDITIONS