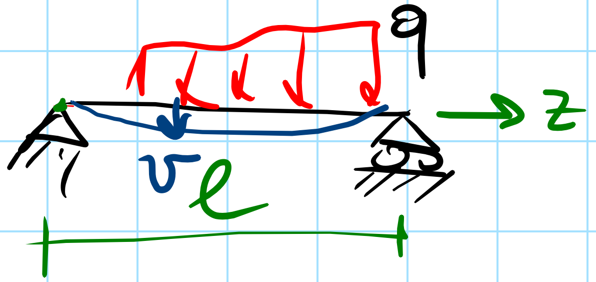
STRONG AND WEAK FORMULATIONS OF A DIFFERENTIAL PROBLEM

CSM, 2/3/26

• THE PROBLEM OF THE "ELASTICA":



$v(z)$: DISPLACEMENT MAP (ELASTIC LINE)

$$(EJ\, v'')'' = q$$

(IV-ORDER DIFF EQ.)

UNKOWN

+ 4 B. CONDITIONS

STRONG FORMULATION OF THE 'ELASTIC LINE' PROBLEM

WE CAN FORMULATE THE SAME PROBLEM THROUGH THE TH. OF VIRTUAL WORK:

$$\int_{STR} M^* K\, dz = \int_{STR} q^* v\, dz$$

CURVATURE ($v'' = K$)

WEAK FORMULATION OF THE 'ELASTIC LINE PROBLEM'

ANOTHER NEW FORMULATION IS THAT BASED ON THE TOTAL POTENTIAL ENERGY

COMPUTATIONAL MECHANICS (LINEAR ELASTICITY)

• COMPUTATIONAL MODEL $\xrightarrow{\text{AIM}}$ $\underset{\sim}{K}\,\underline{u} = \underline{F}$ (LINEAR SYSTEM)

DATA

(APPROXIMATED WITH RESPECT TO THE MATHEMATICAL ELASTIC PROBLEM)

UNKNOWNS (FINITE NUMBER)

PROPERTY OF THE PROBLEM

SOURCES OF APPROXIMATION:

1) FROM THE DOMAIN (DISCR. N° OF NODES)
2) FROM THE "INVOLVED FUNCTIONS" (DESCRIPTION OF DISPLACEMENT / DESCR. OF LOADS)

FROM THE PROBLEM

3) FROM THE NUMERICAL METHOD
4) ... . LIMITED CAPABILITY OF THE CPU

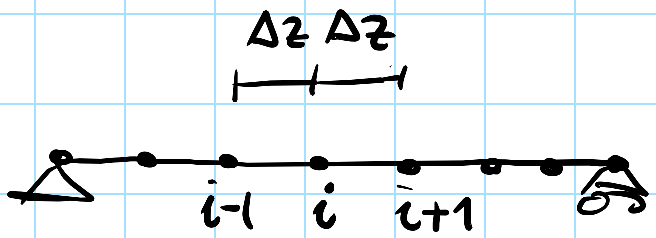
COMPUTATIONAL RESOURCES

REGARDING THE "NUMERICAL METHOD"

1) FINITE DIFFERENCE METHOD (LATE '800)

$(EJ\, v'')'' = q \longrightarrow$ "DISCRETIZE" THE DERIVATIVES

STRONG FORMULATION



$$\frac{\Delta v}{\Delta z}_{(i)} = \frac{v_{i+1} - v_i}{\Delta z}$$

EXAMPLE OF DISCR OF $v'(z)$

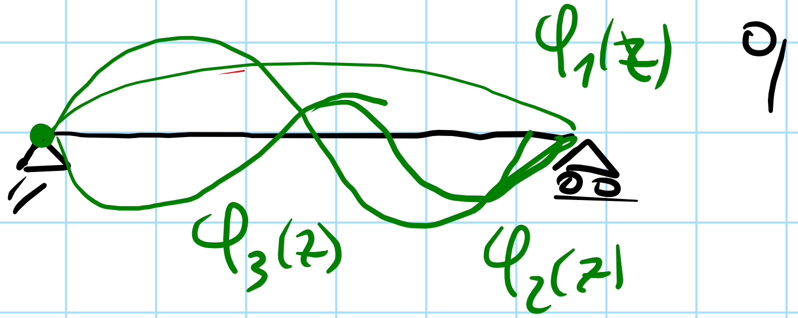
2) RAYLEIGH-RITZ METHOD (LATE '800)

WEAK FORMULATION

$$v(z) = \alpha_1 \varphi_1(z) + \alpha_2 \varphi_2(z) + \alpha_3 \varphi_3(z)$$

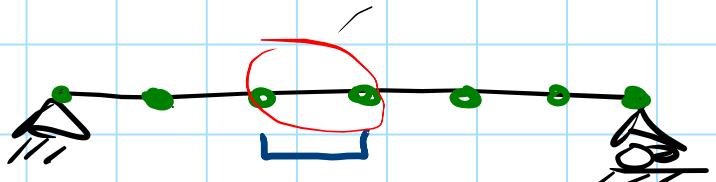
SUITABLE SET OF FUNCTIONS

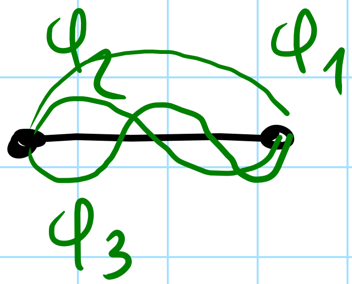
UNKNOWNS OF THE PROBLEM (FINITE NO)
RITZ COEFFICIENTS



3) FINITE ELEMENT METHOD (FEM) (1920 GALERKIN, '60 CLOUGH, HILBERT...)

(extension of RAYLEIGH-RITZ METHOD)





THE DOMAIN IS DIVIDED IN DISCRETE ELEMENTS AND WITHIN EACH ELEMENT THE SOLUTION IS SOUGHT :

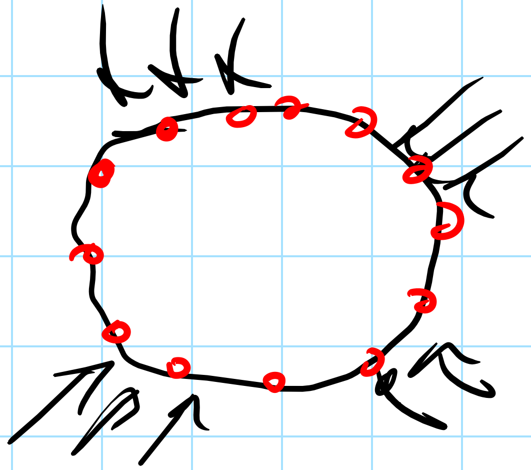
$$u(z) = \alpha_1 \varphi_1(z) + \alpha_2 \varphi_2(z) + \alpha_3 \varphi_3(z) \ldots$$

RITZ COEFFIC. FOR THAT ELEMENT.

THERE ARE TWO TYPES OF APPROXIMATION :

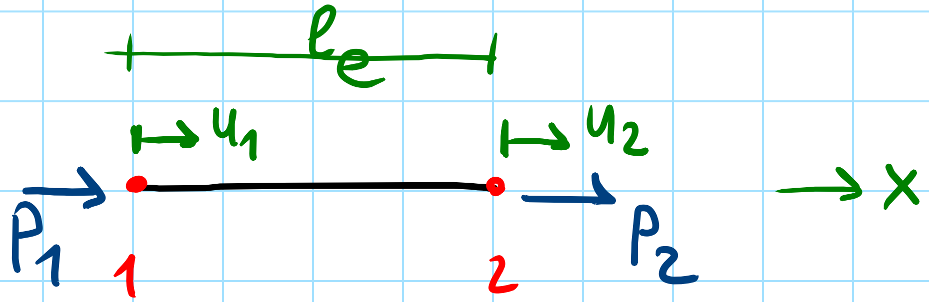
- GEOMETRIC (MESH)
- ANALYTICAL ($\varphi_1, \varphi_2, \varphi_3 \ldots$)

4) BEM : BOUNDARY ELEMENT METHOD



FEM IN ELASTICITY OF BARS

- STIFFNESS MATRIX OF A BAR (EA : CONST)



$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\frac{EA}{\ell_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\underset{\sim}{K}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(SYMMETRIC, SINGULAR)

K_{ij} : FORCE IN i-th NODE WHEN $u_j = 1$ AND $u_i = 0$ $(i \neq j)$

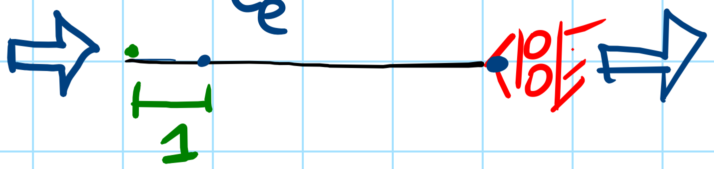
$$\left.\begin{matrix} u_1 = 0 \\ u_2 = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} K_{12} \\ K_{22} \end{matrix}$$

$K_{12} = -\frac{EA}{\ell_e} \cdot 1$

$\Rightarrow K_{22} = \frac{EA}{\ell_e} \cdot 1$

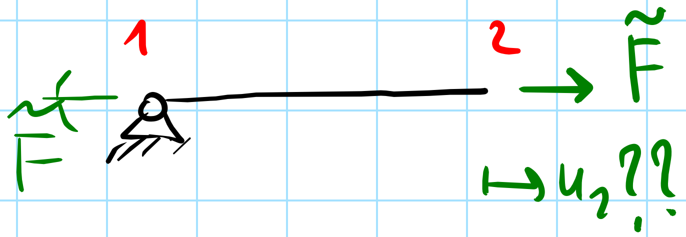
$$\left.\begin{matrix} u_1 = 1 \\ u_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} K_{11} \\ K_{21} \end{matrix}$$

$\Rightarrow K_{11} = \frac{EA}{\ell_e} \cdot 1$



$\Rightarrow K_{21} = -\frac{EA}{\ell_e} \cdot 1$

e structural



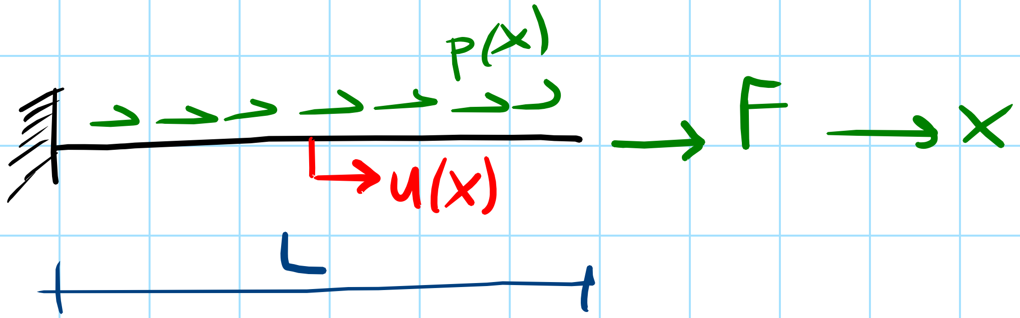
$$\begin{bmatrix} P_1 \\ \tilde{F} \end{bmatrix} = \frac{EA}{\ell_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \end{bmatrix}$$

$$\begin{cases} P_1 = \dfrac{EA}{\ell_e}(-u_2) \\ \boxed{\tilde{F} = \dfrac{EA}{\ell_e} u_2} \end{cases} \Longrightarrow \begin{aligned} &2)\ P_1 = -\tilde{F} \\ &1)\ u_2 = \frac{\tilde{F}\,\ell_e}{EA} \end{aligned}$$

(TWO-STEP SOLUTION)

LET US COME BACK TO OUR GOAL (FEM IN BARS)

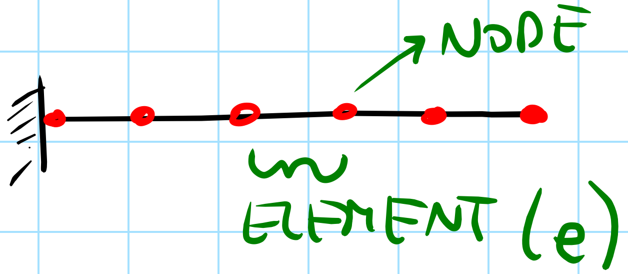
"FEM IN BARS"



WEAK FORMULATION $\rightarrow$ STAT. OF TOT. POT. ENERGY

$$\Pi(u) = \frac{1}{2}\int_0^L EA(u')^2\,dx - \int_0^L p\,u\,dx - F(0)\,u(0) - F(L)\,u(L)$$

1) DISCRETIZATION

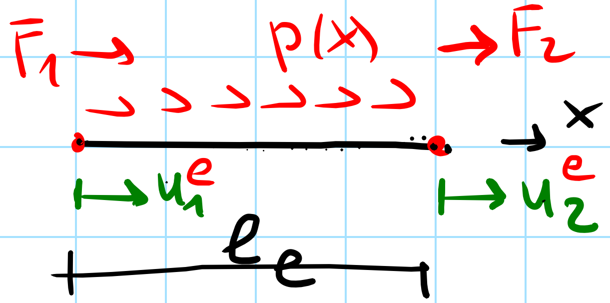


(MESH : m_e ELEMENTS)

$$\Pi(u) \simeq \sum_{e=1}^{m_e} \underbrace{\Pi^e(u^e(x))}_{??}$$

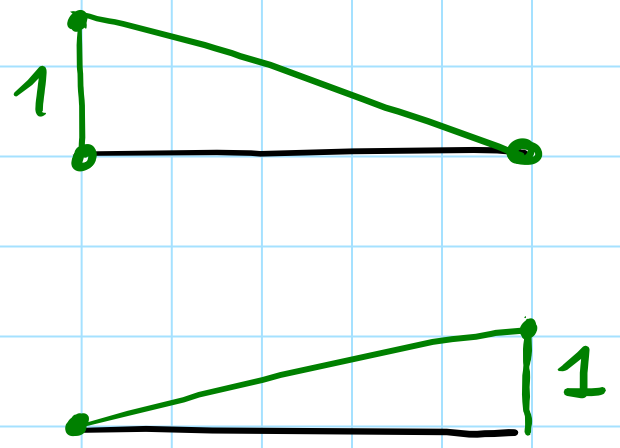
?? $\Rightarrow$ LET US STUDY EACH ELEMENT

2) APPROXIMATION OF THE DISPL. FIELD ($u_e(x)$) IN EACH ELEMENT



ELEMENT e (EA: CONST)

(NODAL DISPLACEMENTS)



$N_1(x)$

$N_2(x)$

$u(x) = \alpha_1 + \alpha_2 x$ (LINEAR COMBINATION) $x \in [0, \ell_e]$

α_1, α_2 → UNKNOWNS (RITZ COEFFS)

$\alpha_1, \alpha_2 = f(u_1, u_2)$?

$$\begin{cases} u(0) = u_1 = \alpha_1 \\ u(\ell_e) = u_2 = \alpha_1 + \alpha_2 \ell_e \end{cases}$$

$$\boxed{\begin{aligned} \alpha_1 &= u_1 \\ \alpha_2 &= \frac{u_2 - u_1}{\ell_e} \end{aligned}}$$

THEREFORE

SHAPE FUNCTION

$$u(x) = u_1 + \frac{u_2 - u_1}{\ell_e} x = u_1 \underbrace{\left(1 - \frac{x}{\ell_e}\right)}_{N_1(x)} + u_2 \underbrace{\left(\frac{x}{\ell_e}\right)}_{N_2(x)} = \underbrace{\left[N_1(x) \quad N_2(x)\right]}_{\underset{\sim}{N}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\underline{U}} = \underset{\sim}{N}\,\underline{U}$$

SCALAR PRODUCT

$$u(x) \longrightarrow \varepsilon(x) = u'(x) = \frac{d}{dx}\left(u_1 N_1(x) + u_2 N_2(x)\right) = u_1 N_1'(x) + u_2 N_2'(x)$$

$$= u_1\left(-\frac{1}{\ell_e}\right) + u_2\left(\frac{1}{\ell_e}\right) = \underbrace{\begin{bmatrix} -\frac{1}{\ell_e} & \frac{1}{\ell_e} \end{bmatrix}}_{\underset{\sim}{B}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\underline{U}} = \underset{\sim}{B}\,\underline{U}$$

STRAIN – DISPL MATRIX

3) T. P. E.

$$\Pi^e(u) = \frac{1}{2}\int_0^{\ell_e} EA\,\varepsilon^2\,dx - \int_0^{\ell_e} p\,u - F_1 u_1 - F_2 u_2$$

$$= \frac{1}{2}\int_0^{\ell_e} (\underbrace{\underset{\sim}{B}\,\underline{U}}_{\varepsilon})^T EA\,(\underbrace{\underset{\sim}{B}\,\underline{U}}_{\varepsilon})\,dx - \int_0^{\ell_e} p\,(\underset{\sim}{N}\,\underline{U})\,dx - [u_1 \;\; u_2]\underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}_{\underline{F}}$$

$$= \frac{1}{2}\int_0^{\ell_e} \underline{U}^T \underset{\sim}{B}^T EA\,\underset{\sim}{B}\,\underline{U}\,dx - \int_0^{\ell_e} \underline{U}^T \underset{\sim}{N}^T p\,dx - \underline{U}^T \underline{F}$$

$$= \frac{1}{2}\,\underline{U}^T \underbrace{\left(\int_0^{\ell_e} \underset{\sim}{B}^T \underset{\sim}{B}\,EA\,dx\right)}_{\underset{\sim}{K}^e} \underline{U} - \underline{U}^T\left(\int_0^{\ell_e} \underset{\sim}{N}^T p\,dx\right) - \underline{U}^T \underline{F}$$

$$= \frac{1}{2}\,\underline{U}^T \underset{\sim}{K}^e \underline{U} - \underline{U}^T \underbrace{\left[\int_0^{\ell_e} \underset{\sim}{N}^T p\,dx - \underline{F}\right]}_{\text{VECTOR OF NODAL FORCES } (\underline{F}^e)} \quad = \quad \Pi^e(\underline{u}) = \frac{1}{2}\,\underline{U}^T \underset{\sim}{K}^e \underline{U} - \underline{U}^T \underline{F}^e$$

$$x - \frac{x^2}{2l_e}$$

$$\underset{\sim}{K}^e = EA \int_0^{l_e} \underset{\sim}{B}^T \underset{\sim}{B} \, dx = EA \int_0^{l_e} \begin{bmatrix} -1/l_e \\ 1/l_e \end{bmatrix} \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix} dx$$

$$= \frac{EA}{l_e^2} \int_0^{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \boxed{\frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underset{\sim}{K}^e}$$

STIFFNESS MATRIX OF A GENERIC ELEMENT

(WITH LINEAR SHAPE FUNCTION)

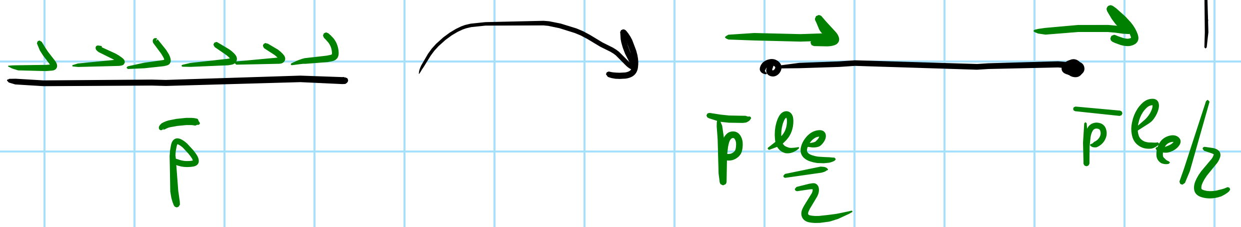
ONLY DISTR. FORCES

$$\underline{F}^e = \int_0^{l_e} p \, \underset{\sim}{N} \, dx = \int_0^{l_e} \underset{\sim}{N}^T p \, dx = \int_0^{l_e} \begin{bmatrix} \left(1 - \frac{x}{l_e}\right) p(x) \\ \frac{x}{l_e} \, p(x) \end{bmatrix} dx$$

IF $p(x)$: CONST $= \bar{p}$ $\Rightarrow$

$$= \bar{p} \int_0^{l_e} \begin{bmatrix} 1 - x/l_e \\ \frac{x}{l_e} \end{bmatrix} dx = \bar{p} \begin{bmatrix} l_e/2 \\ l_e/2 \end{bmatrix}$$

$$\underline{F}^e = \begin{bmatrix} \bar{p} \, l_e/2 \\ \bar{p} \, l_e/2 \end{bmatrix}$$



The general vector

$$\underline{F}^e = \begin{bmatrix} \int_0^{l_e} \left(1 - \frac{x}{l_e}\right) p(x) \, dx + F_1 \\ \int_0^{l_e} \frac{x}{l_e} \, p(x) \, dx + F_2 \end{bmatrix}$$

Up to know $\Pi^e(\underline{U}) = \frac{1}{2}\underline{U}^T \underset{\sim}{K}^e \underline{U} - \underline{U}^T \underline{F}^e$

T.P.E. STATIONARY $\frac{\partial \Pi^e}{\partial \underline{U}} = \underline{0} \Longrightarrow \underset{\sim}{K}^e \underline{U} - \underline{F}^e = \underline{0} \Longrightarrow \boxed{\underset{\sim}{K}^e \underline{U} = \underline{F}^e}$