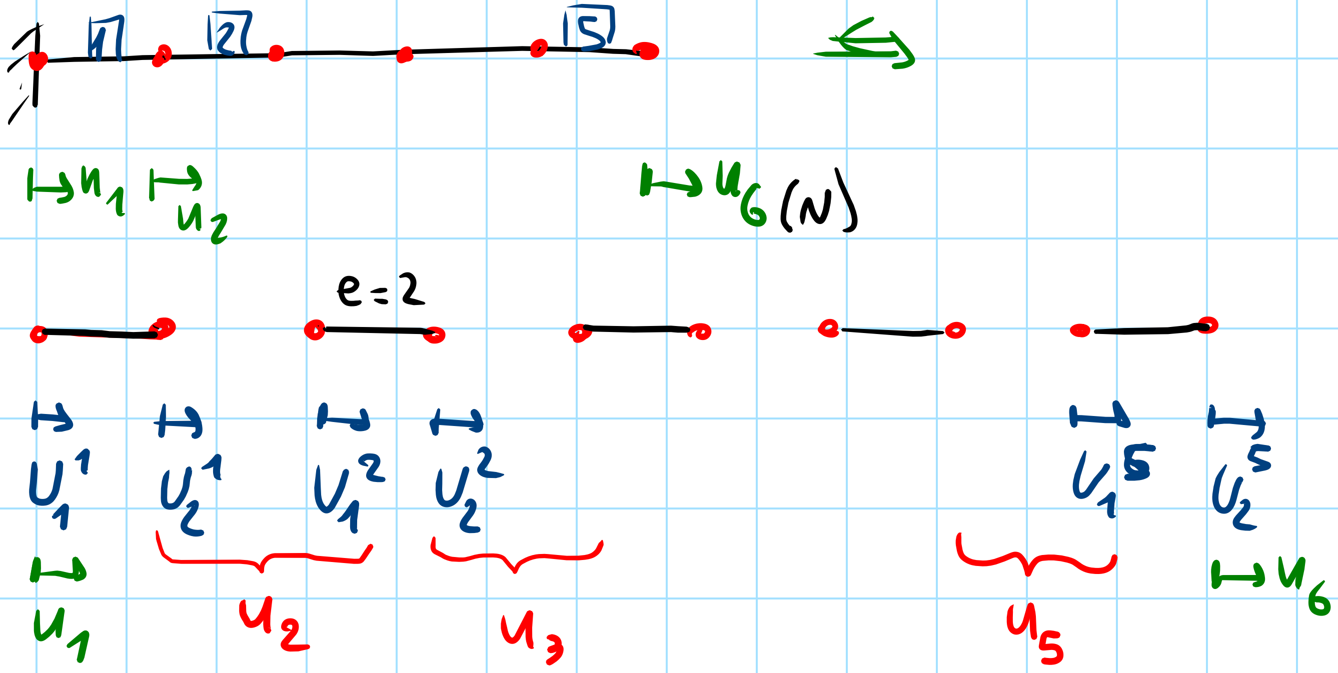
ASSEMBLY PROCEDURES IN A STRUCTURE COMPOSED OF BAR ELEMENTS

CSM, 3/03/26



IN A "LOCAL" ELEMENT $\underset{\sim}{K}^e \underline{U}^e = \underline{F}^e$

FOR THE WHOLE PROBLEM WE NEED TO REACH THE SYSTEM: $\underset{\sim}{K}\, \underline{u} = \underline{F}$

$\underline{u}$

$$\begin{bmatrix} & & \\ & 6\times 6 & \\ & \underset{\sim}{K} & \end{bmatrix} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}}_{\underline{u}} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_6 \end{bmatrix}$$

WE FOLLOW THE "EXPANDED MATRIX METHOD"

$\underset{\sim}{K}^e \underline{U}^e = \underline{F}^e \quad (*)$ $\qquad \underset{\sim}{K}^e : 2\times 2, \quad \underline{U}^e = 2\times 1 = \underline{F}^e$

REWRITE (*) AS $\quad \hat{\underset{\sim}{K}}^e \underline{u} = \hat{\underline{F}}^e, \quad \hat{\underset{\sim}{K}}^e : 6\times 6, \quad \underline{u} : 6\times 1 : \hat{\underline{F}}_e$

ex: take e = 2

NON-NULL COMPONENTS

$$\hat{\underset{\sim}{K}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \bullet & \bullet & 0 & \\ 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} \\ F^e_2 \\ F_3 \\ \\ \end{bmatrix}$$

$$\underline{U}^e = \underset{\sim}{C}^e \underline{u}, \qquad \underset{\sim}{C}^{e=2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

↳ CONNECTIVITY

$$\begin{bmatrix} U_1^2 \\ U_2^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{\underset{\sim}{C}^2} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_6 \end{bmatrix}$$

WRITE THE T.P.E FOR ELEMENT e :

$$\Pi^e(\underline{U}) = \frac{1}{2}\, \underline{U}_e^T \underset{\sim}{K}_e \underline{u}_e - \underline{U}^T \underline{F}_e = \frac{1}{2}\left(\underset{\sim}{C}^e \underline{u}\right)^T \underset{\sim}{K}_e \underset{\sim}{C}^e \underline{u} - \left(\underset{\sim}{C}^e \underline{u}\right)^T \underline{F}_e$$

$$= \frac{1}{2}\, \underline{u}^T \underbrace{\underset{\sim}{C}^{eT} \underset{\sim}{K}_e \underset{\sim}{C}^e}_{\hat{\underset{\sim}{K}}_e} \underline{u} - \underline{u}^T \underbrace{\underset{\sim}{C}^{eT} \underline{F}_e}_{\hat{\underline{F}}_e \text{ (6 ELEMENTS)}}$$

6×6 MATRIX : $(6\times 2)(2\times 2)(2\times 6) = 6\times 6$

$$\Pi(\underline{u}) = \sum_{e=1}^{6} \Pi^e = \sum_{e=1}^{6} \frac{1}{2}\, \underline{u}^T \hat{\underset{\sim}{K}}_e \underline{u} - \underline{u}^T \hat{\underline{F}}_e = \frac{1}{2}\, \underline{u}^T \underbrace{\left(\sum_{e=1}^{6} \hat{\underset{\sim}{K}}_e\right)}_{\underset{\sim}{K}} \underline{u} - \underline{u}^T \underbrace{\left(\sum_{e=1}^{6} \hat{\underline{F}}_e\right)}_{\underline{F}}$$

$\underset{\sim}{K}$: GLOBAL MATRIX

$\underline{F}$: GLOBAL VECTOR OF FORCES

- BECAUSE IN $\underset{\sim}{K}\,\underline{u} = \underline{F}$ THE CONSTRAINTS ARE NOT IMPOSED YET $\longrightarrow$ $\underset{\sim}{K}$ IS SEMI-DEFINITE

$$\underline{u} = \begin{bmatrix} \underline{u}_U \\ \hline \underline{u}_F \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} \underline{F}_U \\ \hline \underline{F}_F \end{bmatrix}$$

$\underline{u}_U, \underline{F}_U$: DISPL. AND FORCES WHERE THE DISPLACEMENTS ARE KNOWN

$\underline{u}_F, \underline{F}_F$: " " " WHERE THE FORCES ARE KNOWN

$$\left[\begin{array}{c|c} \underset{\sim}{K}_{UU} & \underset{\sim}{K}_{UF} \\ \hline \underset{\sim}{K}_{FU} & \underset{\sim}{K}_{FF} \end{array}\right] \begin{bmatrix} \underline{u}_U \\ \hline \underline{u}_F \end{bmatrix} = \begin{bmatrix} \underline{F}_U \\ \hline \underline{F}_F \end{bmatrix}$$

REACTIONS

UNKOWNS OF THE PROBLEM

1) 2ND ROW:

$$\underbrace{\underset{\sim}{K}_{FF}}_{\text{KNOWN}} \underline{u}_F = \underbrace{\underline{F}_F - \underset{\sim}{K}_{FU} \underline{u}_U}_{\text{DATA: KNOWN ELEMENTS}}$$

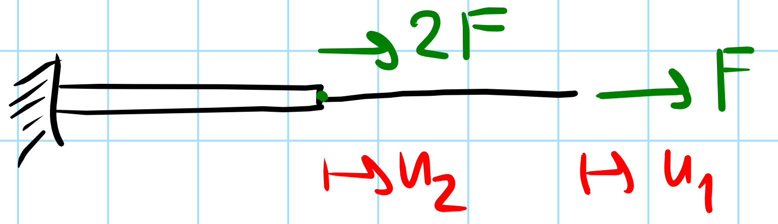
$\underline{u}_F \rightarrow$ UNKOWNS (DISPLACEMENTS)

2) 1ST ROW

$$\underbrace{\underline{F}_U}_{\text{REACTIONS}} = \underbrace{\underset{\sim}{K}_{UU} \underline{u}_U + \underset{\sim}{K}_{UF} \underline{u}_F}_{\text{KNOWN}}$$

QUESTION:

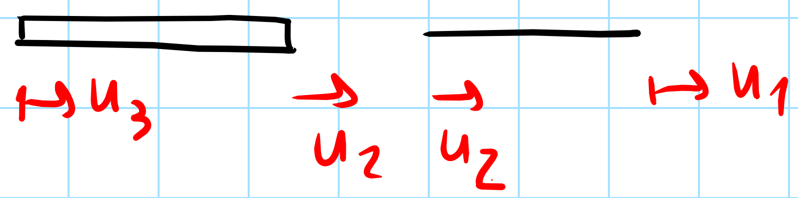
YOU SOLVE BY HAND (WITH DISPL. METHOD) THE PROBLEM



$\underset{\sim}{K}^* \underline{u} = \underline{F}$

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 2F \end{bmatrix}.$$

NOW YOU WANT TO DISCR. BY FEM



3 D.O.F. $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

WHAT ARE $\underline{u}_U$ AND $\underline{u}_F$?

$$\underline{u} = \begin{bmatrix} \underline{u}_U \\ \hline \underline{u}_F \end{bmatrix} = \begin{bmatrix} u_3 = 0 \\ \hline u_1 \\ u_2 \end{bmatrix} \quad , \quad \underline{F} = \begin{bmatrix} \underline{F}_U \\ \hline \underline{F}_F \end{bmatrix} = \begin{bmatrix} F_1 \\ \hline F \\ 2F \end{bmatrix}$$

REACTION

$$\left[\begin{array}{c|c} K_{UU} & \underset{\sim}{K}_{UF} \\ \hline \underset{\sim}{K}_{FU} & \underset{\sim}{K}_{FF} \end{array}\right] \begin{bmatrix} 0 \\ \hline u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ \hline F \\ 2F \end{bmatrix}$$

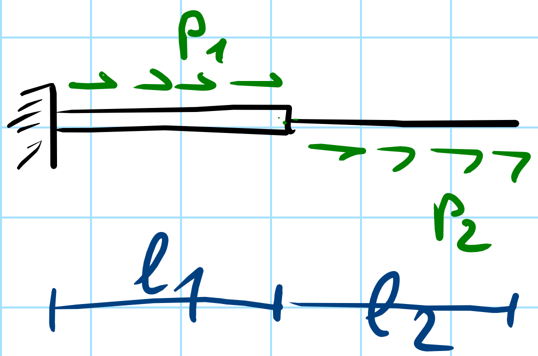
UNKNOWNS

NOTE THAT THE COMPONENTS OF THE STIFFNESS MATRIX K^* (PREVIOUS PAGE) ARE IN $\underset{\sim}{K}_{FF}$

1) $\underset{\sim}{K}_{FF} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 2F \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underset{\sim}{K}_{FF}^{-1} \begin{bmatrix} F \\ 2F \end{bmatrix}$ OBTAIN u_1, u_2

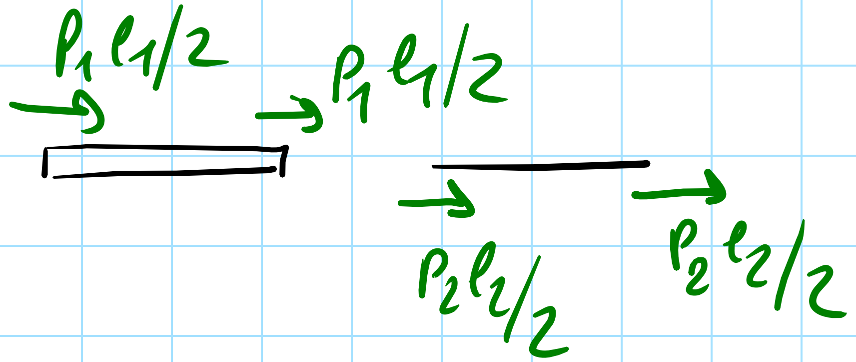
2) $\underset{\sim}{K}_{UF} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = F_1$ OBTAIN F_1

NOTE: NODAL FORCES?



FEM

FOR LINEAR SHAPE FUNCTIONS



QUADRATIC SHAPE FUNCTIONS

USING LINEAR SHAPE FUNCTIONS MEANS $u(x)$ IS LINEAR BUT THE AXIAL FORCE ($N = EA\,u'(x)$) IS PIECEWISE CONSTANT (LINEARE A TRATTI)

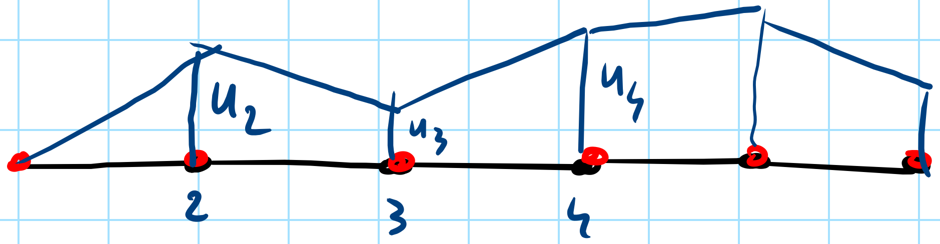
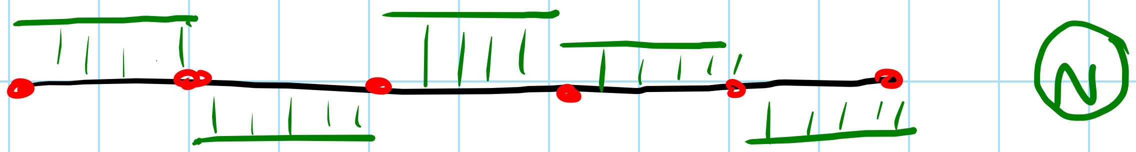
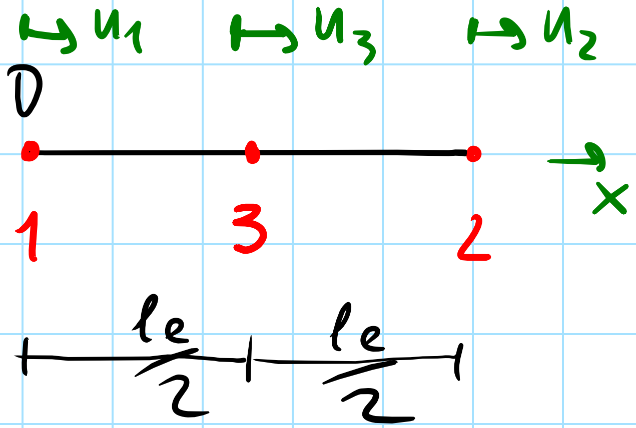


DIAGRAM OF DISPL IN THE WHOLE STRUCTURE (LINEAR SHAPE FUNCTION)

WITH QUADRATIC SHAPE FUNCTIONS WE WANT TO INCREASE THE ACCURACY OF $u(x) \Rightarrow N(x)$ WILL BE LINEAR



N IS IN GENERAL DISCONTINUOUS AT THE NODES



$$u(x) = a + bx + cx^2$$

3 COEFFICIENTS (3 NODES)

$$\begin{cases} u(0) = a = u_1 \\ u\left(\frac{l_e}{2}\right) = a + b\frac{l_e}{2} + c\left(\frac{l_e}{2}\right)^2 = u_3 \\ u(l_e) = a + b\, l_e + c\, l_e^2 = u_2 \end{cases}$$

$$\begin{cases} a = u_1 \\ b = \frac{1}{l_e}\left(-3u_1 - u_2 + 4u_3\right) \\ c = \frac{1}{l_e^2}\left(2u_1 + 2u_2 - 4u_3\right) \end{cases}$$

$$\underline{U}^e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u(x) = \underbrace{\left[1 - \frac{3x}{l_e} + \frac{2x^2}{l_e^2}\right]}_{N_1(x)} u_1 + \underbrace{\left[-\frac{x}{l_e} + \frac{2x^2}{l_e^2}\right]}_{N_2(x)} u_2 + \underbrace{\left[\frac{4}{l_e}x - \frac{4}{l_e^2}x^2\right]}_{N_3(x)} u_3$$

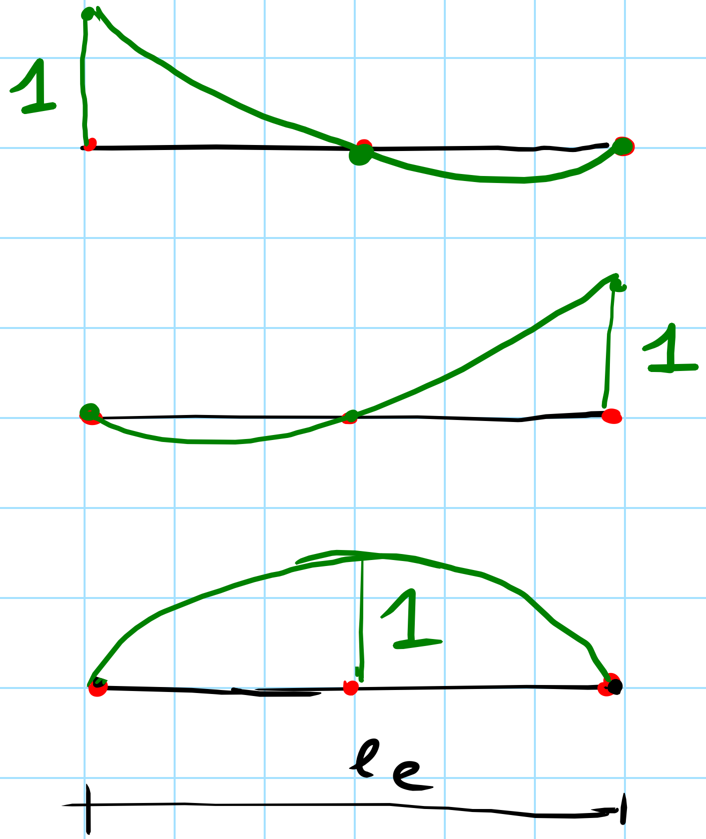
$$u(x) = N_1(x)\, u_1 + N_2(x)\, u_2 + N_3(x)\, u_3 = \underset{\sim}{N}\, \underline{U}^e$$

WHAT ARE THE PLOTS OF $N_i(x)$?

$$N_1(x) = 1 - \frac{3x}{\ell_e} + \frac{2x^2}{\ell_e^2}$$

$$N_2(x) = \frac{x}{\ell_e}\left[-1 + \frac{2x}{\ell_e}\right]$$

$$N_3(x) = 4\frac{x}{\ell_e}\left[1 - \frac{x}{\ell_e}\right]$$



RECALL THAT $\varepsilon_e(x) = u'(x) = \sum_{i=1}^{3} N_i'(x)\, u_i = \underset{\sim}{B}\, \underline{U}^e$

$$\underset{\sim}{B} = \left[\left(-\frac{3}{\ell_e} + \frac{4x}{\ell_e^2}\right) \quad \left(-\frac{1}{\ell_e} + \frac{4x}{\ell_e^2}\right) \quad \left(\frac{4}{\ell_e} - \frac{8x}{\ell_e^2}\right)\right]$$

NOTE AGAIN THAT

$$\underset{\sim}{B}^T \underset{\sim}{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \Big[\ 3 \times 3\ \Big]$$

T.P.E. -

$$\Pi^e(\underline{U}^e) = \frac{1}{2}\, \underline{U}^{eT} \underbrace{\left(\int_0^{\ell_e} \underset{\sim}{B}^T(x)\, EA\, \underset{\sim}{B}(x)\, dx\right)}_{\underset{\sim}{K}^e} \underline{U}^e - \underline{U}^{eT} \underbrace{\left(\int_0^{e} \underset{\sim}{N}^T(x)\, p(x)\, dx - \begin{bmatrix} F_1^e \\ F_2^e \\ F_3^e \end{bmatrix}\right)}_{\underline{F}^e}$$

VECTOR OF NODAL FORCES

$\underset{\sim}{K}^e$ STIFFNESS MAT OF ELEMENT

$\underline{F}^e$

$$\underset{\sim}{K}^e = EA \int_0^{\ell_e} \underset{\sim}{B}^T(x)\,\underset{\sim}{B}(x)\,dx = EA \int_0^{\ell_e} \begin{bmatrix} B_1^2 & B_1B_2 & B_1B_3 \\ & B_2^2 & B_2B_3 \\ & & B_3^2 \end{bmatrix} dx$$

$$K_{11} \overset{?}{=} EA \int_0^{\ell_e} \left(-\frac{3}{\ell_e} + \frac{4x}{\ell_e^2}\right)^2 dx = EA \int \left(\frac{9}{\ell_e^2} + \frac{16x^2}{\ell_e^4} - \frac{24x}{\ell_e^3}\right) dx$$

$$= \frac{EA}{\ell_e^2}\left[9x + \frac{16}{\ell_e^2}\frac{x^3}{3} - \frac{24}{\ell_e}\frac{x^2}{2}\right]_0^{\ell_e} = \frac{EA}{\ell_e^2}\left[9\ell_e + \frac{16}{3}\ell_e - 12\ell_e\right] = \frac{7}{3}\frac{EA}{\ell_e^2}$$

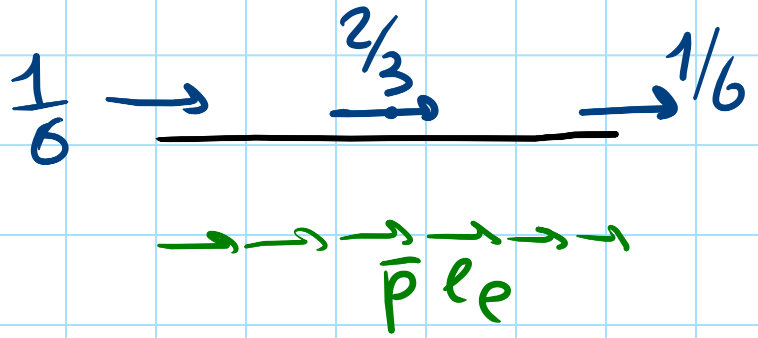
$$\underset{\sim}{K}^e = \frac{EA}{3\ell_e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \quad (3\times 3)$$

WHAT ABOUT NODAL FORCES

FOR $p(x) = \bar{p}$ CONST ?

$$\bar{p}\int_0^{\ell_e} \underset{\sim}{N}^T(x)\,dx = \begin{bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{bmatrix}$$

$$\begin{bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{bmatrix} = \frac{\bar{p}\ell_e}{3}\begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$$

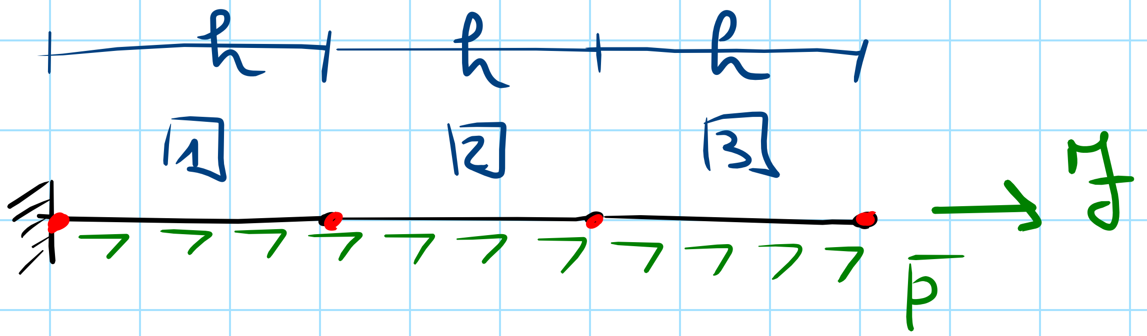


$$Q_1^e \overset{?}{=} \bar{p}\int_0^{\ell_e} N_1(x)\,dx = \bar{p}\int_0^{\ell_e} 1 - \frac{3x}{\ell_e} + \frac{2x^2}{\ell_e^2}\,dx$$

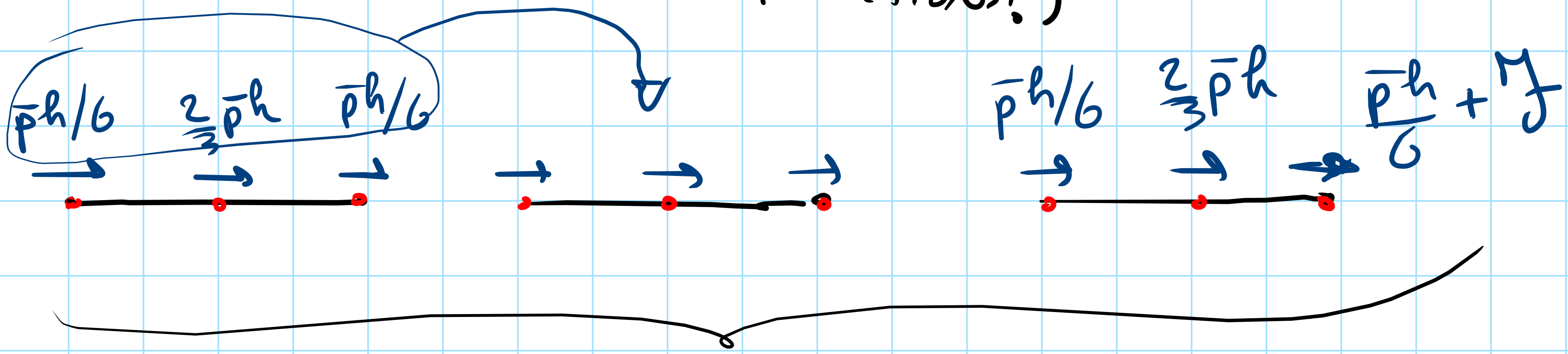
$$= \bar{p}\left[x - \frac{3}{\ell_e}\frac{x^2}{2} + \frac{2}{\ell_e^2}\frac{x^3}{3}\right]_0^{\ell_e} = \bar{p}\ell_e\left[1 - \frac{3}{2} + \frac{2}{3}\right] = \bar{p}\ell_e\,\frac{6-9+4}{6}$$

$$= \frac{1}{6}\bar{p}\ell_e$$

EX

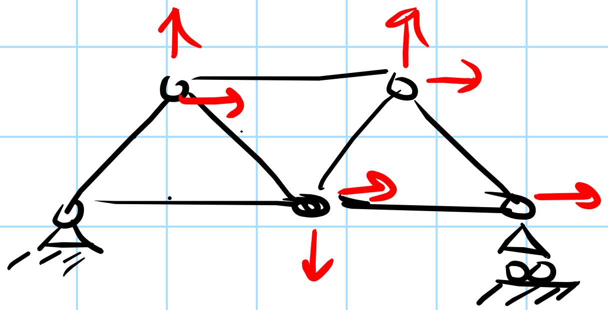


HOW TO SUBDIVIDE THE GIVEN LOADS IN THE 3 ELEMENTS (QUADRATIC SHAPE FUNCTIONS!)

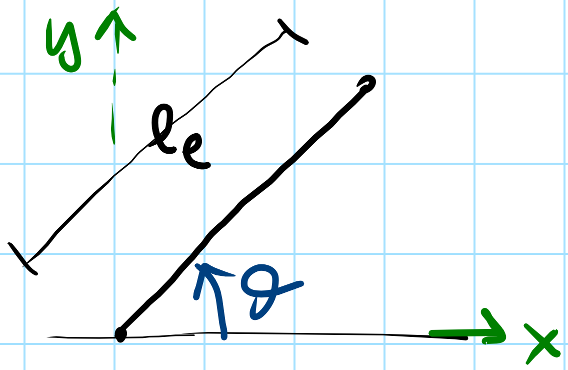


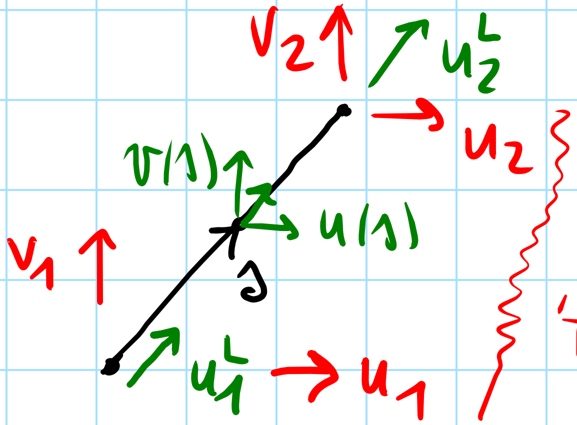
WE KNOW NOW THE VECTOR OF NODAL FORCES OF EACH ELEMENT.

HOW TO DEAL WITH INCLINED ELEMENTS



(LINEAR SHAPE FUNCTIONS)





D.O.F. OF "THE STRUCTURE"

$$\underline{U}^e = \begin{bmatrix} u_1^L \\ u_2^L \end{bmatrix}$$

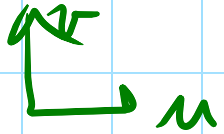
WHAT IS THEIR RELATIONSHIP?

$$\underline{U} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

OUR GOAL

$$\begin{cases} u(s) = N_1(s)\, u_1 + N_2(s)\, u_2 \\ v(s) = N_1(s)\, v_1 + N_2(s)\, v_2 \end{cases}$$

TWO COMPONENTS OF THE AXIAL DISPLACEMENT IN THE SYSTEM

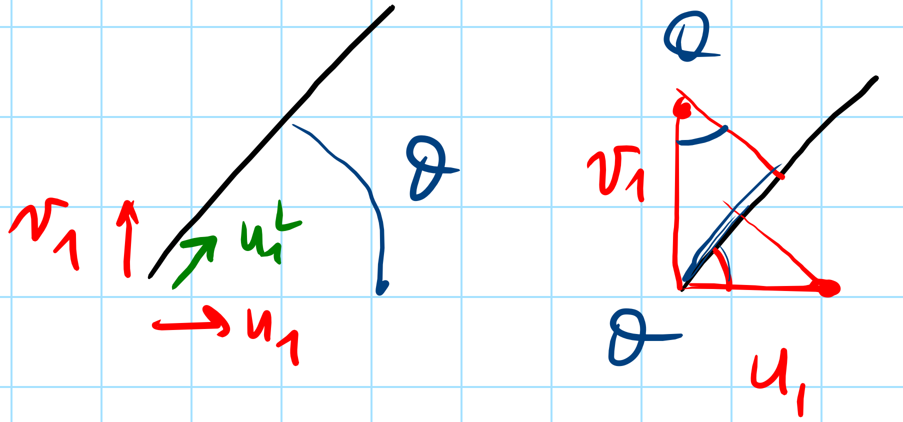


$$\begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = \begin{bmatrix} N_1(s) & 0 & N_2(s) & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

→ THE STIFFNESS MATRIX OF THE ELEMENT IN THE NEW REFERENCE SYSTEM CAN BE OBTAINED FROM THIS FORMULATION CONTINUING WITH THE T.P.E. AND INTEGRATE ....

ALTERNATIVELY, WE OBSERVE THAT THERE IS A DIRECT APPROACH TO OBTAIN OUR RELATIONSHIP:

$$\underbrace{\begin{bmatrix} u_1^L \\ u_2^L \end{bmatrix}}_{\underline{U}^e} = \underbrace{\begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 & 0 \\ 0 & 0 & \cos\vartheta & \sin\vartheta \end{bmatrix}}_{\underset{\sim}{T}\,(2\times4)} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}}_{\underline{U}}$$



FOR THE SINGLE ELEMENT:

$$\Pi^e = \frac{1}{2}\underline{U}^{eT}\underset{\sim}{K}_e\underline{U}^e - \underline{U}^{eT}\underline{F}_e = \frac{1}{2}(\underset{\sim}{T}\underline{U})^T\underset{\sim}{K}_e\underset{\sim}{T}\underline{U} - (\underset{\sim}{T}\underline{U})^T\underline{F}_e$$

$$= \frac{1}{2}\underline{U}^T\underbrace{\underset{\sim}{T}^T\underset{\sim}{K}_e\underset{\sim}{T}}_{\underset{\sim}{K}\,(4\times4)}\underline{U} - \underline{U}^T\underbrace{\underset{\sim}{T}\underline{F}_e}_{\underline{F}}$$

VECTOR OF NODAL FORCES IN THE NEW REF SYSTEM

STIFFNESS MATRIX OF THE ELEMENT IN THE NEW REF SYSTEM

$$\underline{U} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_1 \end{bmatrix};$$

WITHIN $\underset{\sim}{K}$ THERE ARE EA, l_e AND SEVERAL PRODUCTS OF $\sin\vartheta\cos\vartheta$