... ROTATION OF A BAR

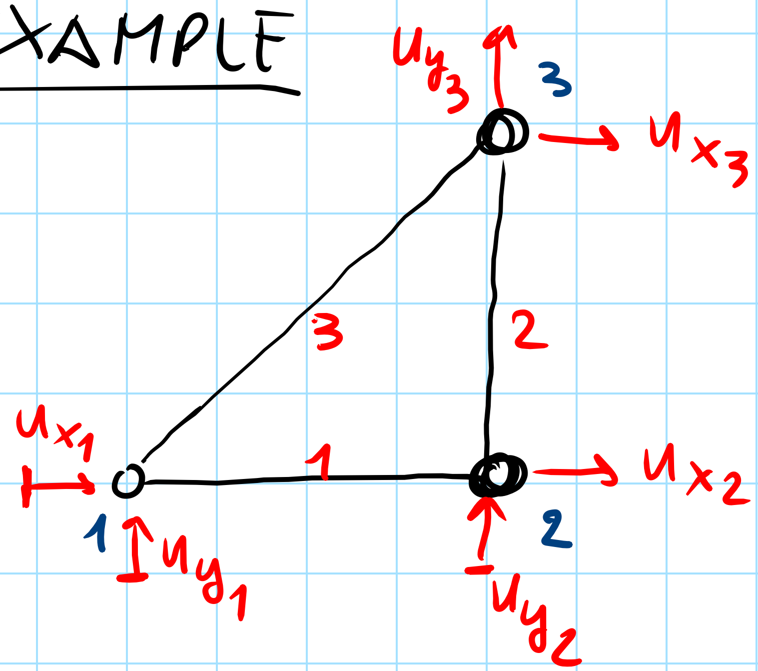
$$\boxed{\begin{array}{l}\cos\theta = C\\ \sin\theta = S\end{array}}$$

CSM, 9/3/26

$$\underset{\sim}{K} = \underset{\sim}{T}^T \underset{\sim}{K}^e \underset{\sim}{T} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \frac{EA}{l_e} = \frac{EA}{l_e} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

STILL FOR THE ELEMENT

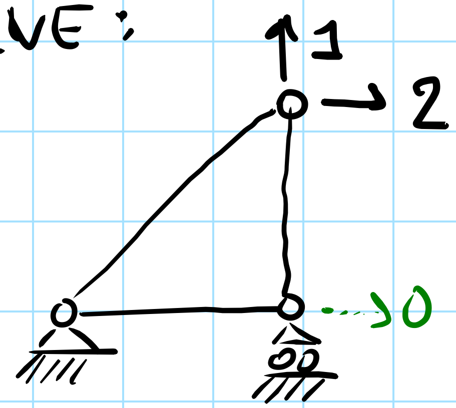
EXAMPLE



$u_{x_1}, u_{y_1}, \dots, u_{x_3}, u_{y_3}$: GLOBAL D.O.F.S OF THE PROBLEM

6

WITH THIS ASSEMBLY SOLVE:



APPLIED FORCES

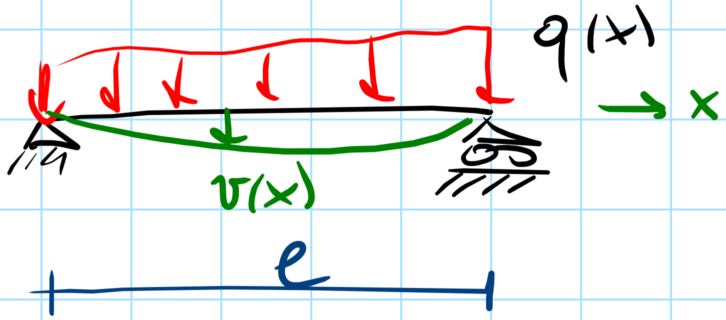
SEE PROVIDED NOTES TO FOLLOW THE PROCEDURE TO GET ALL ELEMENTS OF $\underset{\sim}{K}$

$$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F}$$

$$\begin{bmatrix} & & \\ & \underset{\sim}{K} & \\ & & 6\times 6 \\ & & \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ \vdots \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ \vdots \\ f_{y_3} \end{bmatrix}$$

6

F.E. FORMULATION FOR EULER–BERNOULLI BEAMS



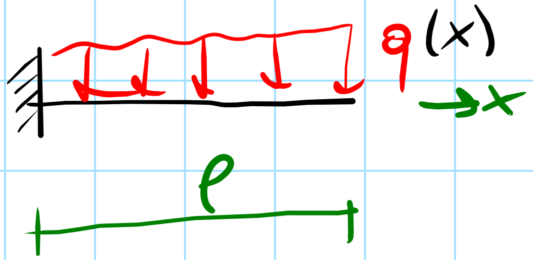
EJ: CONST

• GOVERNING EQ: $\boxed{EJ\, v^{IV}(x) = q(x)}$ $(*)$

• BOUNDARY CONDITIONS (SIMPLY SUPPORTED):

$$\left.\begin{aligned} &v(0) = v(\ell) = 0 \\ &-EJ v''(0) = 0 \\ &-EJ v''(\ell) = 0 \end{aligned}\right\}$$

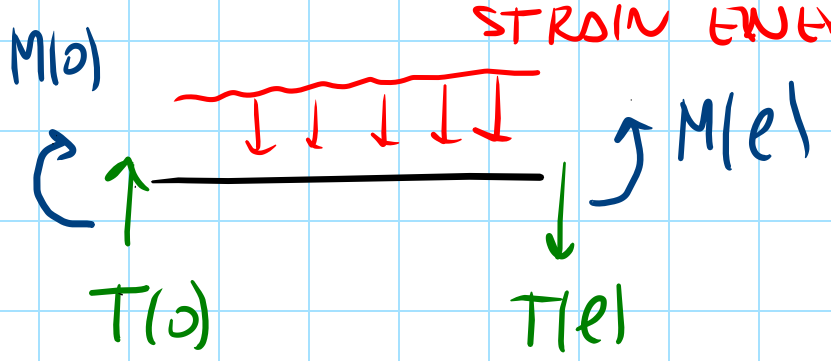
• " (CANTILEVER)

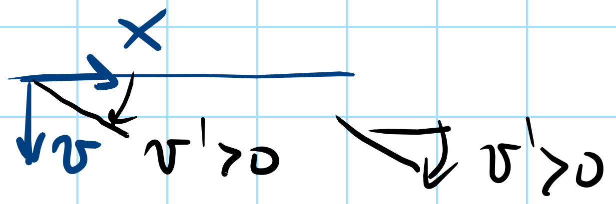


$$\left.\begin{aligned} &v(0) = v'(0) = 0 \\ &-EJ v''(\ell) = 0 \\ &-EJ v'''(\ell) = 0 \end{aligned}\right\}$$

T. P. E. FOR AN E-B BEAM

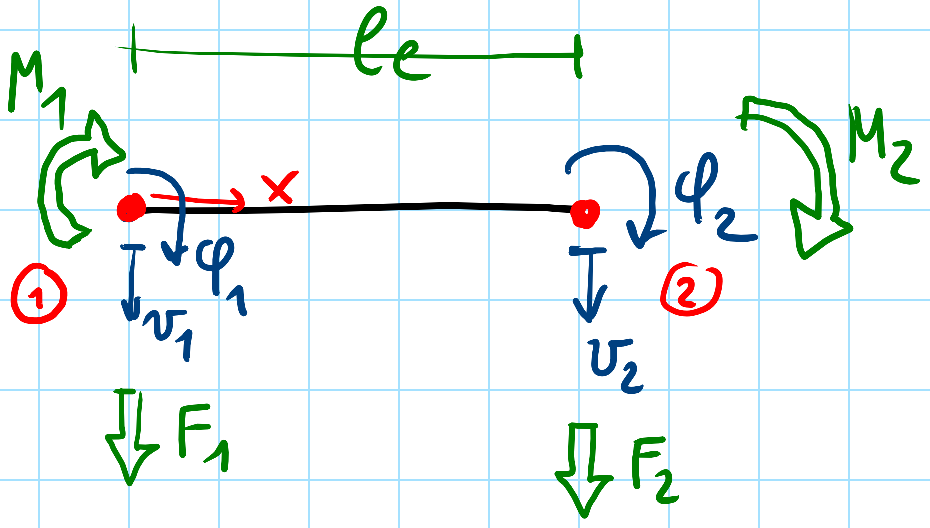
$$\Pi(v) = \underbrace{\frac{1}{2}\int_0^\ell EJ\,(v'')^2\,dx}_{\text{STRAIN ENERGY}} \underbrace{- \int_0^\ell q\,v\,dx - \big(-T(0)\,v(0)\big) - M(0)\,v'(0) - T(\ell)\,v(\ell) - \big(-M(\ell)\,v'(\ell)\big)}_{\text{WORK OF EXTERNAL LOADS}}$$





IF WE STUDY $\boxed{\delta\Pi(v)=0}$ $\rightsquigarrow$ WE OBTAIN EQ. $(*)$

F.E.M. FOR AN E-B. BEAM



4 D.O.F.s : $v_1, \varphi_1, v_2, \varphi_2$

NODAL FORCES: 4 : F_1, M_1, F_2, M_2

WITH THE 4 D.O.Fs WE WOULD LIKE TO MODEL THE FUNCTIONS:

$v(x) \in [0, l_e]$

$v'(x) \in [0, l_e]$

THE POLYN. WITH MINIM. DEGREE IS x^3;

$$v(x) = a + bx + cx^2 + dx^3$$

RITZ-GALERKIN COEFFICIENTS

TO OBTAIN "THE COEFFICIENTS" WE IMPOSE 4 CONDITIONS AT THE NODES (2 IN NODE 1, 2 " " 2)

NODE 1 : v_1, φ_1

NODE 2 : v_2, φ_2

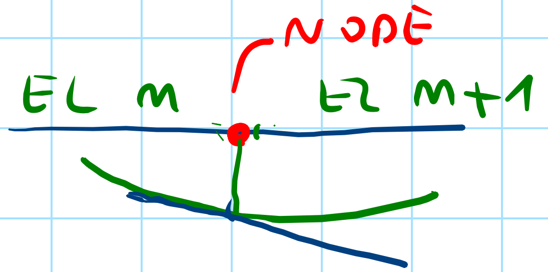
$$\begin{cases} v_1 = v(0) = a \\ \varphi_1 = v'(0) = b \\ v_2 = v(l_e) = a + b l_e + c l_e^2 + d l_e^3 \\ \varphi_2 = v'(l_e) = b + 2c l_e + 3d l_e^2 \end{cases}$$

The solution:

$$\begin{cases} a = v_1 \\ b = \varphi_1 \\ c = \dfrac{3}{\ell_e^2}(v_2 - v_1) - \dfrac{2\varphi_1}{\ell_e} - \dfrac{\varphi_2}{\ell_e} \\ d = \dfrac{2}{\ell_e^3}(v_1 - v_2) + \dfrac{\varphi_1}{\ell_e^2} + \dfrac{\varphi_2}{\ell_e^2} \end{cases}$$

NOTE THAT, FROM THE KINEMATICAL POINT OF VIEW, AT EACH NODE, NOT ONLY $v(x)$ MUST BE CONTINUOUS, BUT ALSO $v'(x)$!

↳ CONCEPT AT THE BASE OF THE "HERMITIAN" SHAPE FUNCTIONS



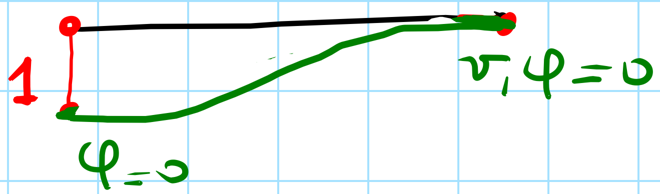
TANGENT IS THE SAME

$$v(x) = v_1 \underbrace{\left(1 - \frac{3x^2}{\ell_e^2} + \frac{2x^3}{\ell_e^3}\right)}_{N_1(x)} + \varphi_1 \underbrace{\left(x - \frac{2x^2}{\ell_e} + \frac{x^3}{\ell_e^2}\right)}_{N_2(x)} + v_2 \underbrace{\left(\frac{3x^2}{\ell_e^2} - \frac{2x^3}{\ell_e^3}\right)}_{N_3(x)} + \varphi_2 \underbrace{\left(-\frac{x^2}{\ell_e} + \frac{x^3}{\ell_e^2}\right)}_{N_4(x)}$$

↳ HERMITIAN SHAPE FUNCTIONS

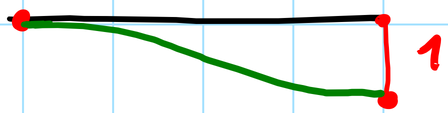
GRAPHS OF $N_i(x)$

$N_1(x)$ $[v_1]$:

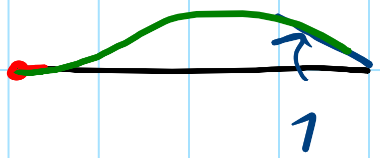


$N_2(x)$ $[\varphi_1]$:

$N_3(x)$ $[v_2]$:

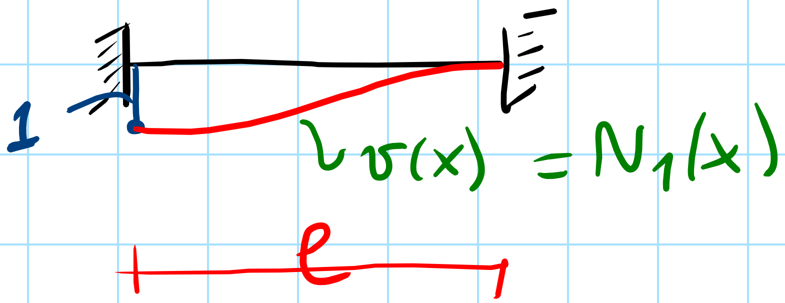


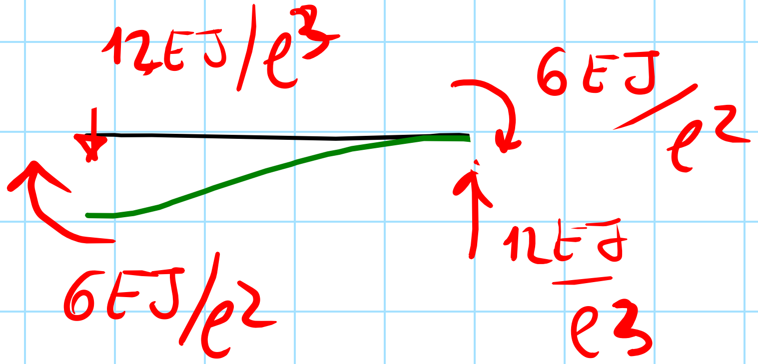
$N_4(x)$ $[\varphi_2]$:



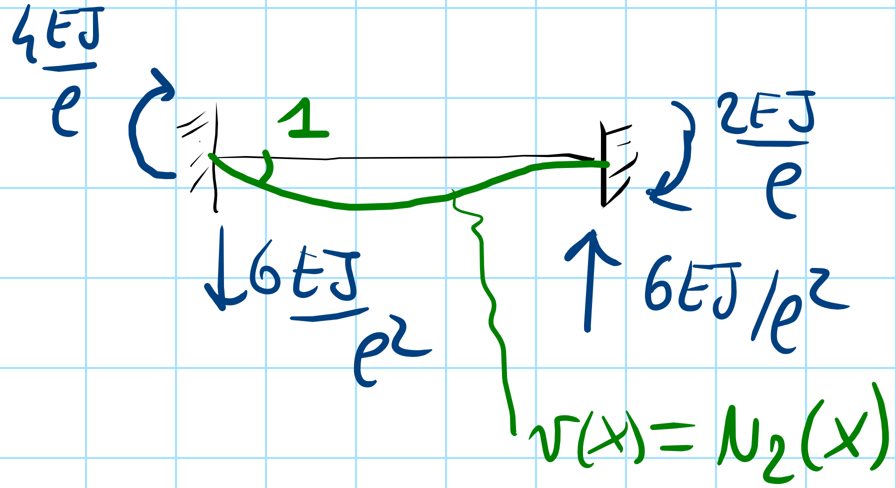
Recall that $N_i(x)$ is $= 1$ at the dof "i" and $= 0$ in all others.

Note that $N_1(x)$ is the deformed beam of the following problem





$N_2(x)$: THE deformed beam of the problem



Note that FORCES/MOMENTS AT THE CONSTRAINTS IN THE 2 ABOVE SCHEMES COINCIDE WITH ELEMENTS OF $\tilde{K}$ (by definition of K_{ij})

Let us write $v(x) = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix}}_{\substack{\text{MATRIX OF SHAPE FUNCTIONS} \\ \underset{\sim}{N}(x)}} \underbrace{\begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix}}_{\substack{\text{VECTOR OF NODAL DOFS} \\ \underline{U}}} = \underset{\sim}{N}\,\underline{U}$

$$v''(x) = \underbrace{\begin{bmatrix} N_1''(x) & N_2''(x) & N_3''(x) & N_4''(x) \end{bmatrix}}_{\underset{\sim}{B}(x)} \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} = \underset{\sim}{B}\,\underline{U}$$

Let us put those in the TOTAL POTENT. ENERGY

$$\Pi(\underline{U}) = \frac{1}{2}\int_0^{l_e} \underline{U}^T \underset{\sim}{B}^T EJ\, \underset{\sim}{B}\,\underline{U}\, dx - \int_0^{l_e} q\, \underline{U}^T \underset{\sim}{N}^T dx - \underline{U}^T \underbrace{\underline{F}}_{\text{CONCENTRATED NODAL FORCES}}$$

$$= \frac{1}{2}\underline{U}^T \underbrace{\left(\int_0^{l_e} \underset{\sim}{B}^T \underset{\sim}{B}\, EJ\, dx\right)}_{\underset{\sim}{K}^e\ (4\times 4)} \underline{U} - \underline{U}^T \underbrace{\left(\int_0^{l_e} q\, \underset{\sim}{N}^T dx + \underline{F}\right)}_{\underline{F}^e\ (4\times 1)} = \frac{1}{2}\underline{U}^T \underset{\sim}{K}\,\underline{U} - \underline{U}^T \underline{F}^e$$

$$\underset{\sim}{B} = \left[\frac{12x}{\ell_e^2} - \frac{6}{\ell_e^2}, \ \frac{6x}{\ell_e^2} - \frac{4}{\ell_e}, \ \frac{6}{\ell_e^2} - \frac{12x}{\ell_e^3}, \ \frac{6x}{\ell_e^2} - \frac{2}{\ell_e} \right]$$

$$\underset{\sim}{K} = \frac{EJ}{\ell_e^3} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ & & 12 & -6\ell_e \\ \text{SYM} & & & 4\ell_e^2 \end{bmatrix}$$

NOTE THAT DIMENSIONS OF K_{ij} ARE NOT ALL THE SAME.

$$K_{12} = EJ \int_0^{\ell_e} B_1 B_2 \, dx = EJ \int_0^{\ell_e} 6 \left(\frac{2x}{\ell_e^2} - \frac{1}{\ell_e^2} \right) \cdot \frac{2}{\ell_e} \left(\frac{3x}{\ell_e} - 2 \right) dx$$

$$= \frac{6EJ}{\ell_e^2}$$

$$\underset{\sim}{K} \underline{U} = \underline{F}$$

2ND LINE :

$$\text{MOMENT } (M_1) = K_{21} v_1 + K_{22} \varphi_1 + K_{23} v_2 + K_{24} \varphi_2$$

$$M_1 = \frac{EJ}{\ell_e^3} \left(6\ell_e v_1 + 4\ell_e^2 \varphi_1 - 6\ell_e v_2 + 2\ell_e^2 \varphi_2 \right)$$

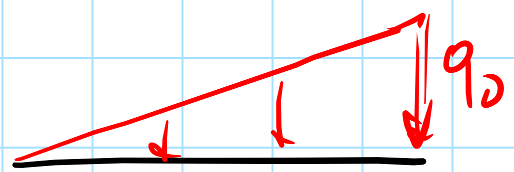
MOMENT : $\alpha \frac{EJ}{\ell} \varphi$, $\beta \frac{EJ}{\ell^2} v$

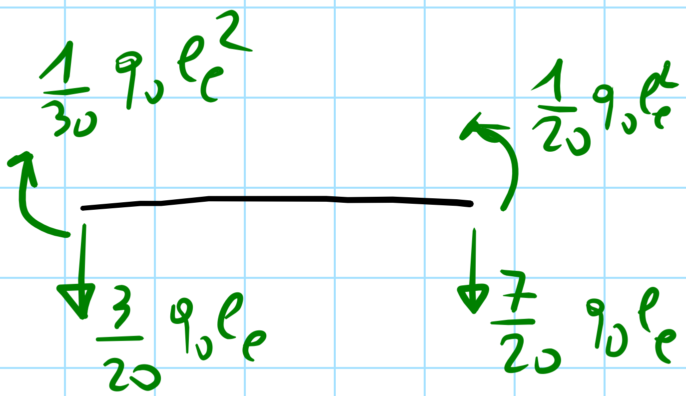
$$\underline{F}^e = \int_0^{l_e} q(x) \begin{bmatrix} N_1(x)\\ N_2(x)\\ N_3(x)\\ N_4(x)\end{bmatrix} dx + \underbrace{\begin{bmatrix} F_1\\ M_1\\ F_2\\ M_2\end{bmatrix}}$$

DISTRIBUTED FORCE ON THE ELEMENT

CONCENTR FORCES AT THE NODES

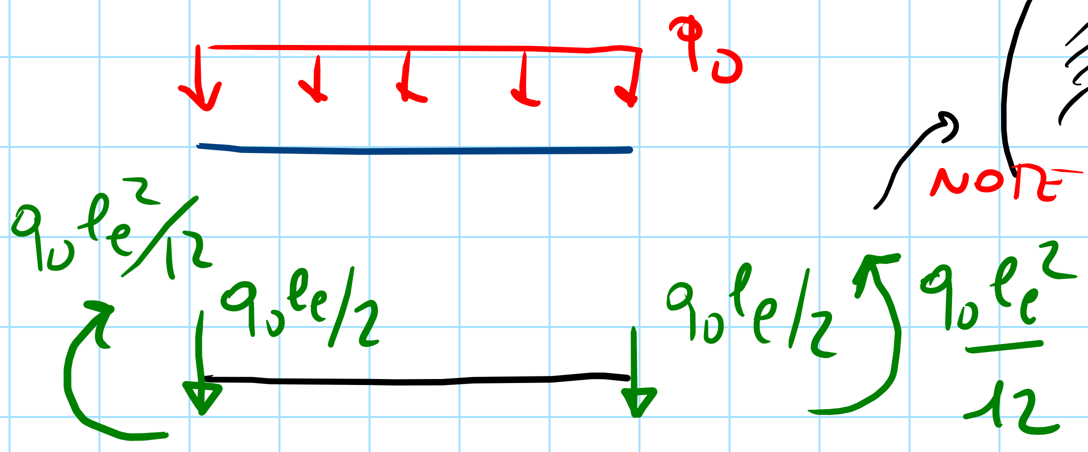
NOTE THAT:

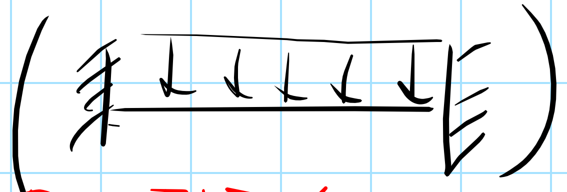




EXAMPLE: COMPUTE $\underline{F}^e$ FOR $q(x)=q_0$ (CONST)

$$\underline{F}^e = q_0 \int_0^{l_e} \begin{bmatrix} N_1(x)\\ N_2(x)\\ N_3(x)\\ N_4(x)\end{bmatrix} dx = \begin{bmatrix} q_0 l_e/2\\ q_0 l_e^2/12\\ q_0 l_e/2\\ -\frac{q_0 l_e^2}{12}\end{bmatrix}$$





NOTE THE CONNECTION WITH THIS PROBLEM

$$\int_0^{l_e} N_4\, dx = \int_0^{l_e} \left(-\frac{x^2}{l_e} + \frac{x^3}{l_e^2}\right) dx = \left[-\frac{x^3}{3l_e} + \frac{x^4}{4l_e^2}\right]_0^{l_e} = -\frac{l_e^2}{3} + \frac{l_e^2}{4} = \frac{-4+3}{12} l_e^2 = -\frac{1}{12} l_e^2$$