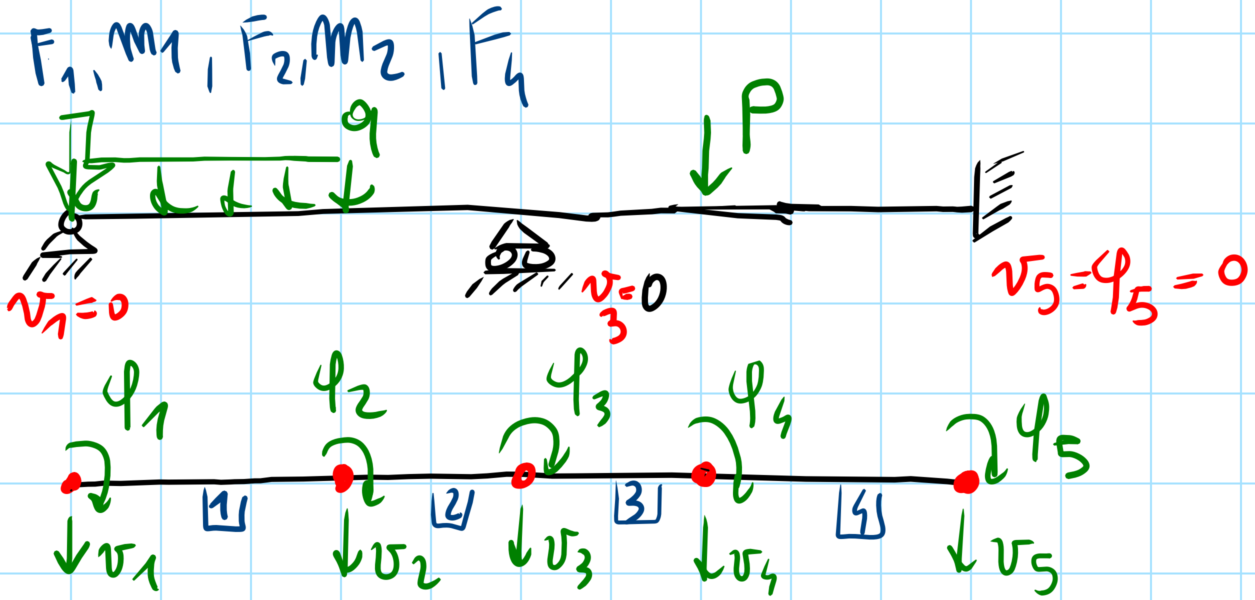
LET US CONTINUE TO WORK WITH BEAM ELEMENTS — CSM, 10/03/26

FROM SIMPLE ELEMENTS TO A BEAM/FRAME PROBLEM

THE GLOBAL SYSTEM WILL BE : $\underbrace{\underset{\sim}{K}}_{\text{STIFFN MATRIX}} \underline{u} = \underline{F}$

$\underline{F}$ ← VECTOR OF GLOBAL NODAL FORCES

$\underline{u}$ ← VECTOR OF GLOBAL DOFs



$$\underbrace{\begin{bmatrix} & & \\ & \underset{\sim}{K} & \\ & & \end{bmatrix}}_{10\times10} \underbrace{\begin{bmatrix} v_1 \\ \varphi_1 \\ \vdots \\ v_5 \\ \varphi_5 \end{bmatrix}}_{10} = \underbrace{\begin{bmatrix} F_1 \\ m_1 \\ \vdots \\ F_5 \\ m_5 \end{bmatrix}}_{10}$$

EACH ELEMENT HAS A LOCAL $\underset{\sim}{K}^e = \frac{EJ}{\ell_e^3}\left[\,4\times4\,\right]$

~~THAT~~ THAT SHOULD BE FIT IN

THE 10x10 $\underset{\sim}{K}$.

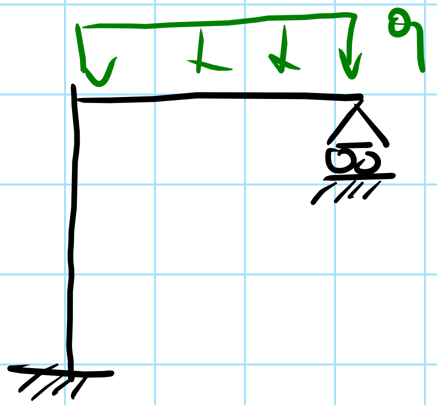
HOWEVER, TO SOLVE THE PROBLEM WE NEED TO PARTITION $\underset{\sim}{K}$ TO DISTINGUISH KNOWN DATA FROM UNKNOWNS.

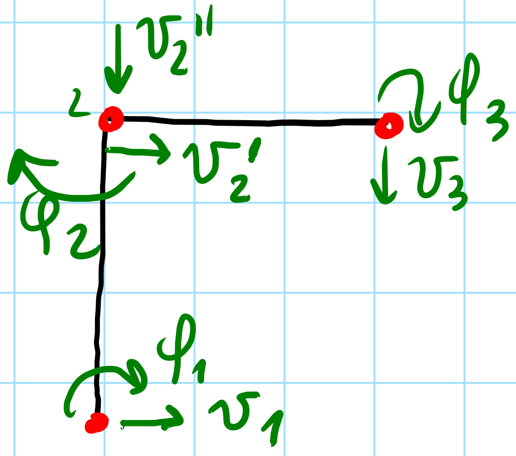
$$\begin{bmatrix} \underset{4\times 4}{K_{UU}} & \underset{4\times 6}{K_{UF}} \\ \underset{6\times 4}{K_{FU}} & \underset{6\times 6}{K_{FF}} \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} v_1 = 0 \\ v_3 = 0 \\ v_5 = 0 \\ \varphi_5 = 0 \end{matrix}\right\} \text{DATA} \\ \hline \varphi_1 \\ v_2 \\ \varphi_2 \\ \varphi_3 \\ v_4 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} F_1 + F_1(q) \\ F_3 \\ F_5 \\ m_5 \\ \hline \left.\begin{matrix} m_1 \\ F_2 \\ m_2 \\ 0 \to m_3 \\ F_4 \quad (P) \\ 0 \to m_4 \end{matrix}\right\} \text{KNOWN (DATA)} \end{bmatrix}$$

→ SOLVE THE SYSTEM IN TWO STEPS.

(SEE A PREVIOUS LECTURE ⇒ 'BAR PROBLEM')

FOR A FRAME THINGS ARE A BIT MORE COMPLEX



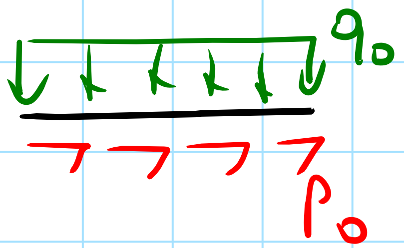


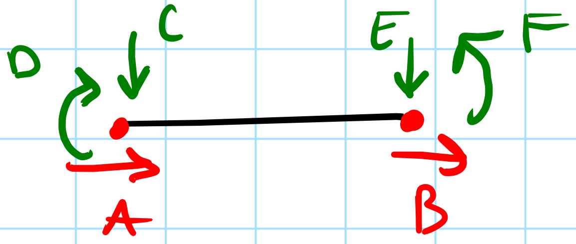
DIFFICULT TO MATCH FLEXURAL BEHAVIOUR / DESCRIPTION TO AXIAL BEHAVIOUR / DESCRIPTION OF ORTHOGONAL ELEMENTS

AND

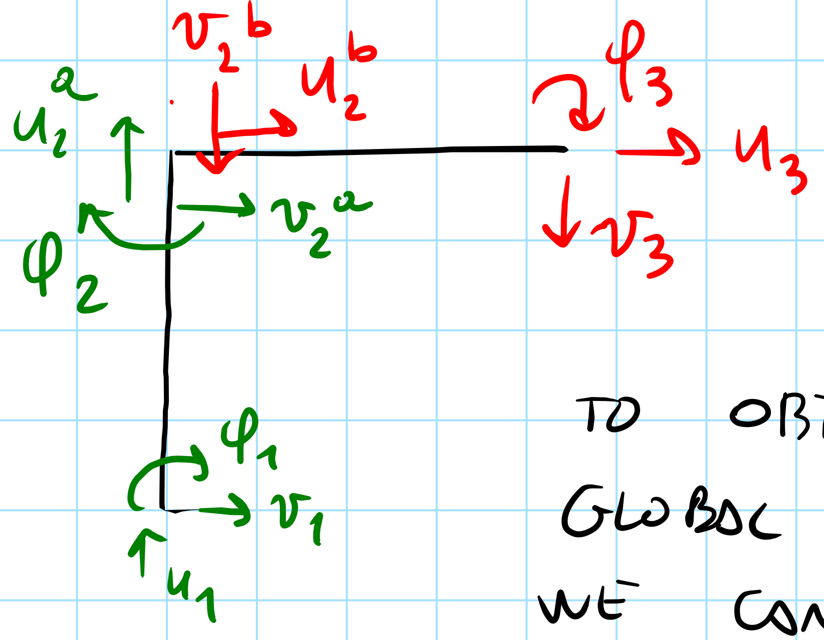
$$\underline{F}^e = \underline{\tilde{F}} + \int_0^{\ell} \underset{\sim}{N}^T \begin{bmatrix} p(x) \\ q(x) \end{bmatrix}$$

FOR CONST $p(x) = p_0$ AND $q(x) = q_0$ AND





$$\Rightarrow \quad \underline{F}^e = \begin{array}{c} \begin{bmatrix} p_0 \ell/2 \\ q_0 \ell/2 \\ q_0 \ell^2/12 \\ p_0 \ell/2 \\ q_0 \ell/2 \\ -q_0 \ell^2/12 \end{bmatrix} \end{array} \begin{array}{c} A \\ C \\ D \\ B \\ E \\ F \end{array}$$

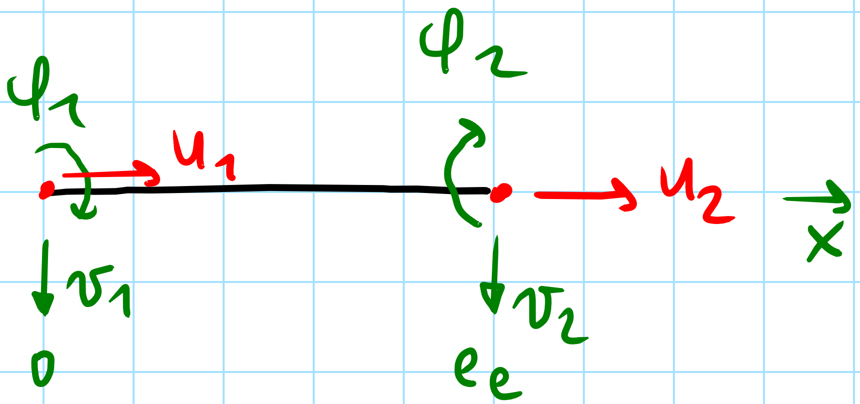


TO OBTAIN THE GLOBAL DOFS WE CAN IMPOSE

$$\begin{cases} u_2^a = -v_2^b \\ v_2^a = u_2^b \end{cases}$$

SO THAT THE N° OF GLOBAL DOFS IS 9

WE CAN THEN CONSIDER A "COMPLETE" EULER–BERNOULLI F.E.



EACH NODE HAS 3 DOFs

$$\underline{U}^e = \begin{bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{bmatrix}$$

HOW THE AXIAL DISPL. $u(x)$ IS DESCRIBED?

$$u(x) = \underbrace{\begin{bmatrix} N_1^u(x) & N_2^u(x) \end{bmatrix}}_{N_i^u(x) \text{ FOR BARS}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

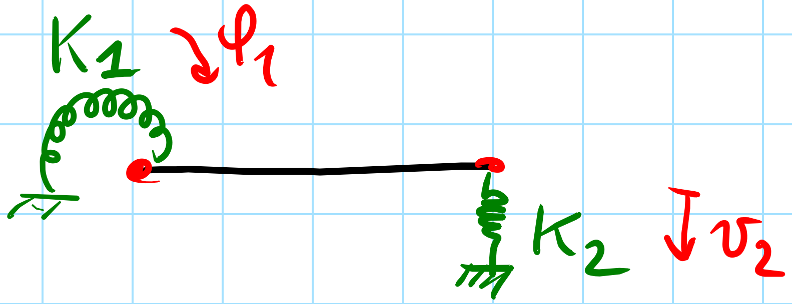
$$v(x) = \begin{bmatrix} N_1^v(x) & N_2^v(x) & N_3^v(x) & N_4^v(x) \end{bmatrix} \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix}$$

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} N_1^u(x) & 0 & 0 & N_1^u(x) & 0 & 0 \\ 0 & N_1^v & N_2^v & 0 & N_3^v & N_4^v \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{bmatrix}$$

WE CAN PUT IN THE T.P.E., INTEGRATE AND OBTAIN

$$\underset{\sim}{K}^e = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & \frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} & 0 & -\frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} \\ 0 & \frac{6EJ}{\ell^2} & \frac{4EJ}{\ell} & 0 & -\frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -\frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} & 0 & \frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} \\ 0 & 6EJ/\ell^2 & 2EJ/\ell & 0 & -6EJ/\ell^2 & 4EJ/\ell \end{bmatrix}$$

HOW TO DEAL WITH GROUND SPRINGS



element level

T.P.E : $\underbrace{\frac{1}{2}\,\underline{U}^T \underset{\sim}{K}_e\, \underline{U}}_{\text{STRAIN ENERGY } [\Phi]} - \underline{U}^T \underline{F}_e$

TO MODIFY STRAIN ENERGY

$$\underline{U} = \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} \qquad \varphi_1 \to U_2, \quad v_2 \to U_3$$

$$\Phi(\underline{U}) = \frac{1}{2}\,\underline{U}^T \underset{\sim}{K}_e\, \underline{U} + \underbrace{\frac{1}{2} K_1\, U_2^{2}}_{(*)} + \underbrace{\frac{1}{2} K_2\, U_3^{2}}_{(**)} = \frac{1}{2}\,\underline{U}^T \underset{\sim}{K}_e\, \underline{U} + \overbrace{\frac{1}{2}\,\underline{U}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{U}}^{(*)} + \overbrace{\frac{1}{2}\,\underline{U}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{U}}^{(**)}$$

($U_2^2 = \varphi_1^2$, $U_3^2 = v_2^2$)

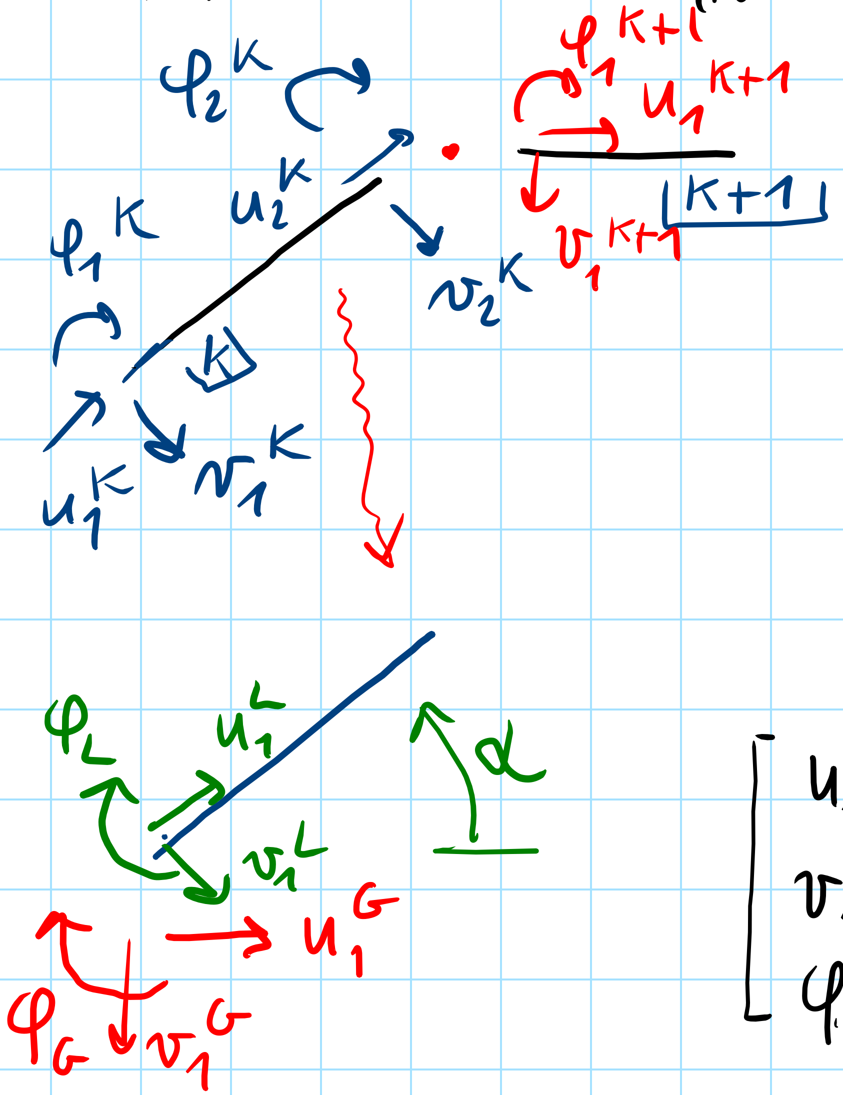
$$= \frac{1}{2}\,\underline{U}^T\, \underset{\sim}{\hat{K}}_e\, \underline{U}$$

$$\underset{\sim}{\hat{K}}_e = \underset{\sim}{K}_e + \begin{bmatrix} 0 & K_1 & 0 \\ 0 & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & 0 \\ & K_2 & \\ 0 & & 0 \end{bmatrix} = \begin{array}{c} \text{USUAL COMPONENTS} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{4EJ}{l_e} + K_1 & \cdot & \cdot \\ \cdot & \cdot & \frac{12EJ}{l_e^3} + K_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

K_1, K_2 ARE ADDED TO THE RELEVANT K_{ij} IN THE DIAGONAL

NOTE THAT K_1, K_2 HAVE DIFF. DIMENSIONS BECAUSE THE SPRINGS ARE DIFFERENT

WHAT'S WHEN INCLINED BEAMS COME INTO PLAY?



FOR AN ELEMENT

$$\begin{bmatrix} u_1^L \\ v_1^L \\ \varphi_1^L \\ u_2^L \\ v_2^L \\ \varphi_2^L \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{T_0} & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{T_0} \end{array}\right] \begin{bmatrix} u_1^G \\ v_1^G \\ \varphi_1^G \\ u_2^G \\ v_2^G \\ \varphi_2^G \end{bmatrix}$$

$$\begin{bmatrix} u_1^L \\ v_1^L \\ \varphi_L \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^G \\ v_1^G \\ \varphi_G \end{bmatrix}$$

$$\underline{u}^L = \underset{\sim}{T_0} \; \underline{u}^G$$

$$\underline{U}^L = \underset{\sim}{T} \; \underline{U}^G$$

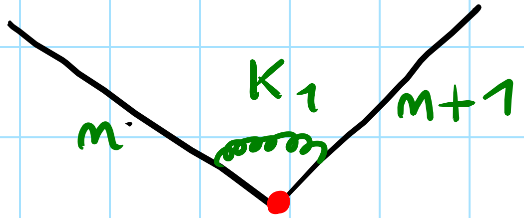
THEREFORE, USING THE T.P.E.

$$\Pi(U^L) = \frac{1}{2} \underline{U}^{LT} \underset{\sim}{K}_e^L \underline{U}^L - \underline{U}^{LT} \underline{F}_e$$

$$\Rightarrow \underset{\sim}{K}_e^G = \underset{\sim}{T}^T \underset{\sim}{K}_e^L \underset{\sim}{T}$$

$$\underline{F}_e^G = \underset{\sim}{T}^T \underline{F}_e$$

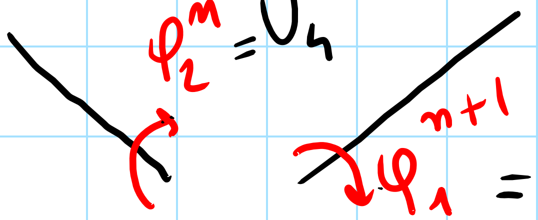
INTERNAL SPRINGS



INTERNAL ROTATIONAL SPRING

IN GENERAL WE ADD TO THE $\Phi(\underline{U})$ A TERM: $\frac{1}{2} K (U_i^m - U_j^{m+1})^2$

IN THE CASE SKETCHED:



$\varphi_2^m = U_4^m$

$\varphi_1^{m+1} = U_2^{m+1}$

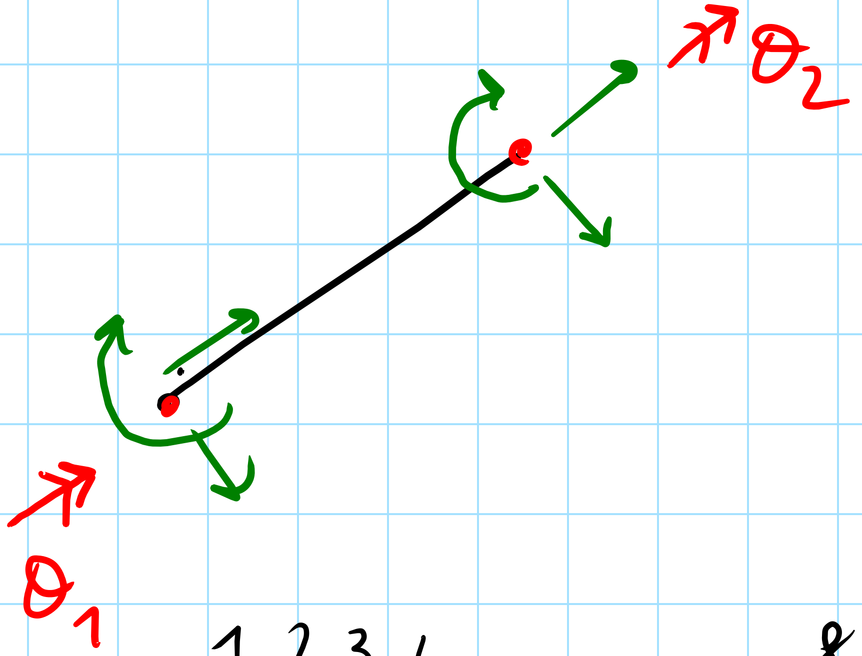
$$\frac{1}{2} K_1 (U_2^{m+1} - U_4^m)^2 = \frac{1}{2} K_1 \left({U_2^{m+1}}^2 + {U_4^m}^2 - 2 U_2^{m+1} U_4^m\right)$$

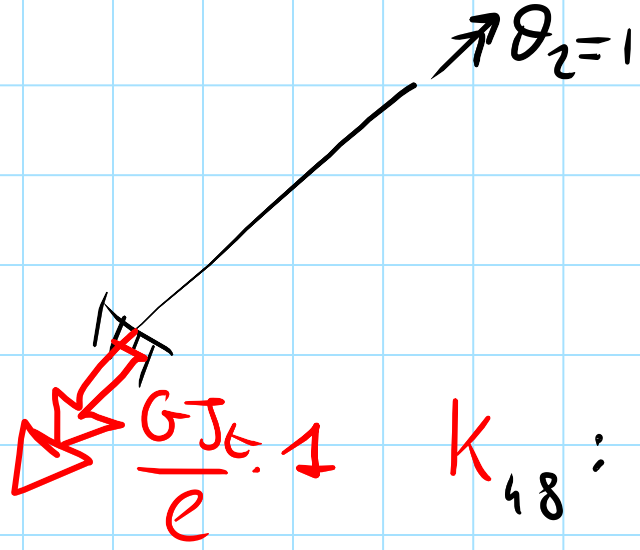
$\Rightarrow$ WE CAN CONSTRUCT 3 ADDIT. STIFFNESS MATRICES WITH ONLY K_1 AS A NON-NULL COMPONENT.

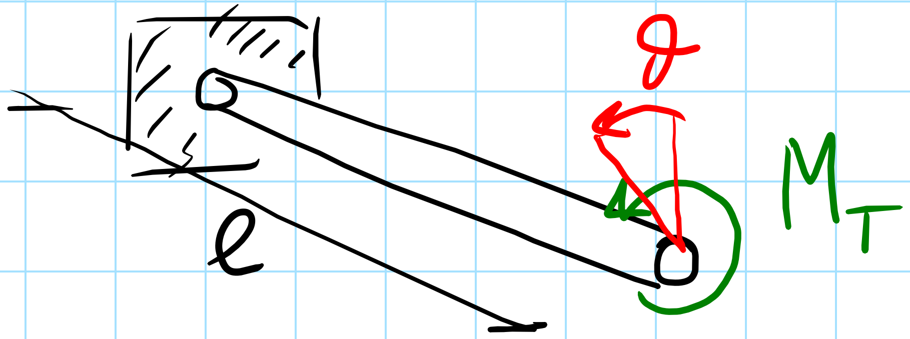
THIS CAN BE DONE NOT IMMEDIATELY AT THE SINGLE ELEMENT LEVEL, BUT IT MUST BE DONE CONSIDERING THE TWO ADJACENT ELEMENTS

POSSIBLE EXTENSIONS:

- PLANE BEAM WITH TORSIONAL STIFFNESS (ADD TWO ROTATIONS ϑ_1, ϑ_2)







K_{48} : TORQUE IN ϑ_1 WHEN $\vartheta_2 = 1$

$$\vartheta = \frac{M_T\, e}{G J_t}$$

G: SHEAR MODULUS

J_t: TORSIONAL MODULUS

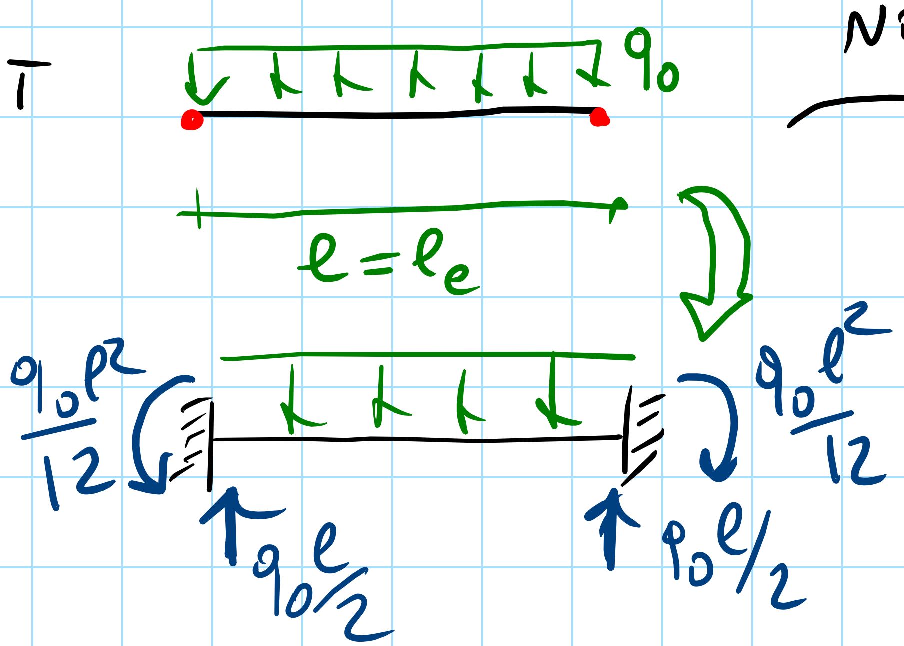
GJ_t: TORS. STIFFN.

$$G = \frac{E}{2(1+\nu)} \qquad J_t = \frac{\pi R^4}{2}$$

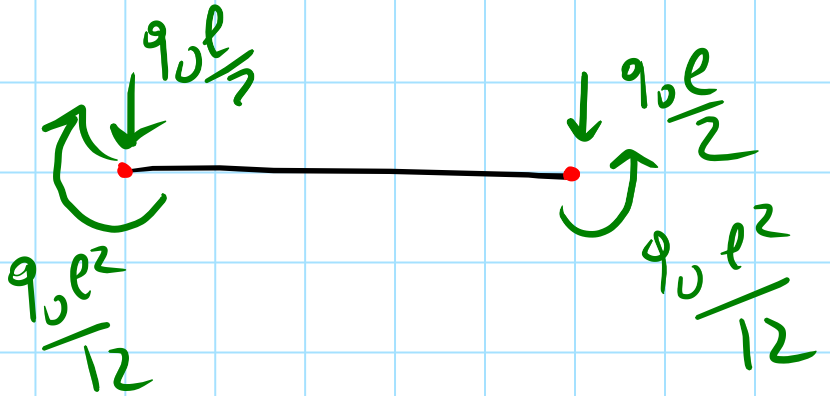
$$\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & & & & 8 \end{array} \\
\hline
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \\ \\ \\ 8 \end{array} &
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
0 & 0 & 0 & \frac{GJ_t}{e} & 0 & 0 & 0 & -\frac{GJ_t}{e} \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
0 & 0 & 0 & -\frac{GJ_t}{e} & 0 & 0 & 0 & \frac{GJ_t}{e}
\end{bmatrix}
\end{array}
\begin{bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ \vartheta_1 \\ u_2 \\ v_2 \\ \varphi_2 \\ \vartheta_2 \end{bmatrix}
=
\begin{bmatrix} \\ \\ \\ M_T^1 \\ \\ \\ \\ M_{T_2} \end{bmatrix}$$

— THERMAL EFFECTS IN BEAMS

CONSIDER FIRST

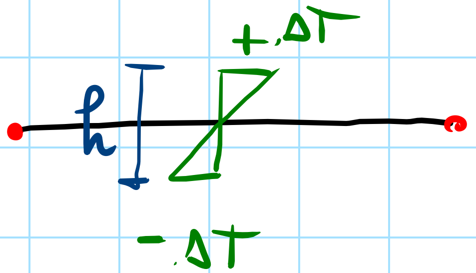


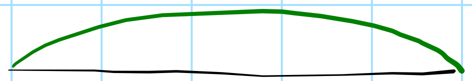
NODAL FORCES

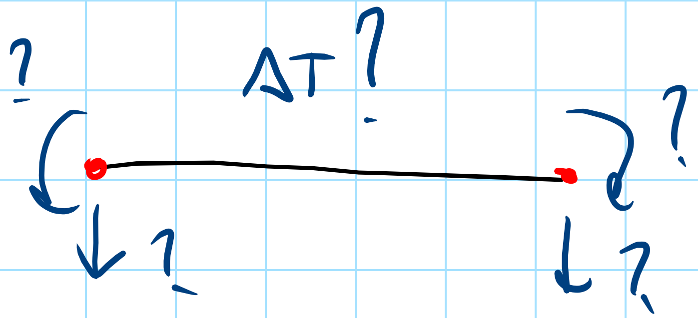


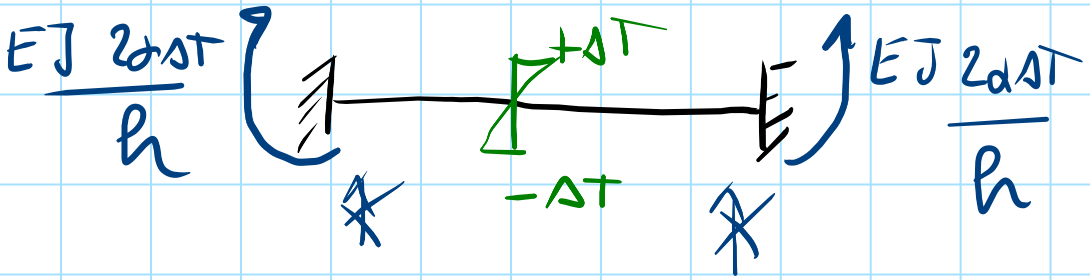
CHANGE SIGNS OF REACTIONS

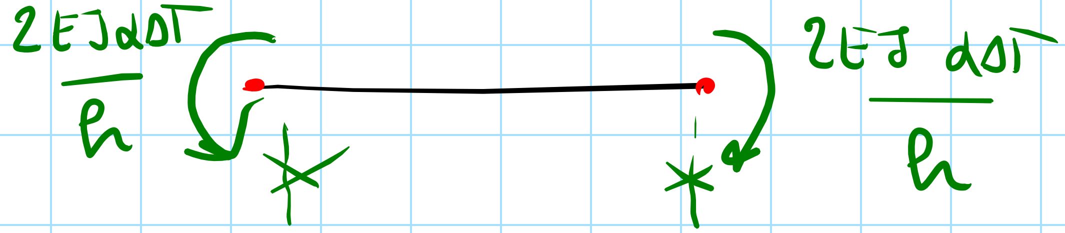
CONSIDER



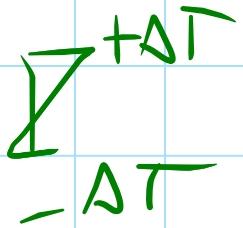




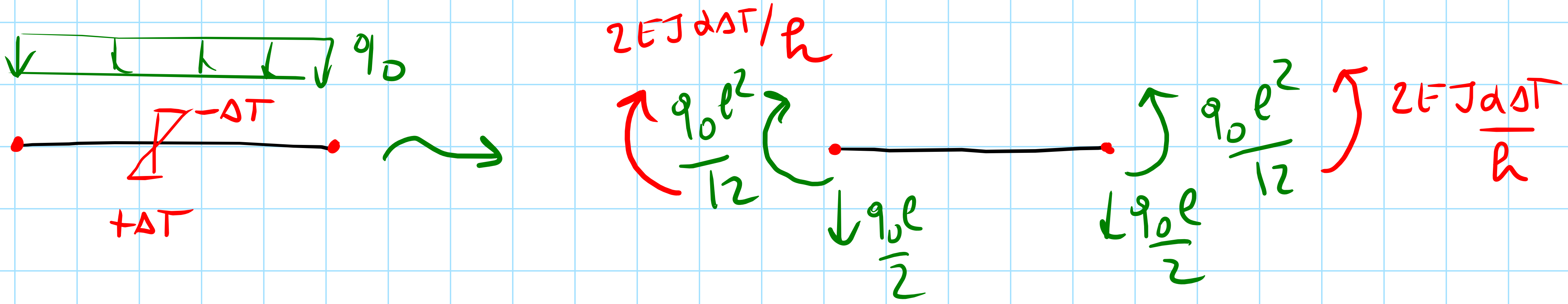




NODAL FORCES FOR A CONST



EX

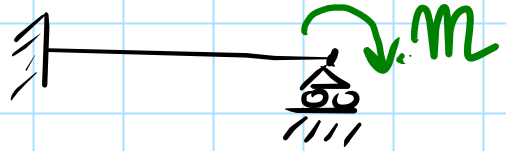


$$\underline{F}_e = \begin{bmatrix} q_0 l/2 \\ \frac{q_0 l^2}{12} + 2EJ\alpha\Delta T/h \\ q_0 l/2 \\ -\frac{q_0 l^2}{12} - \frac{2EJ\alpha\Delta T}{h} \end{bmatrix}$$

NOTE THE FOLLOWING REGARDING THE SOLUTION OF BENDING PROBLEMS WITH FEM.

SHAPE FUNCTIONS are : $a + bx + cx^2 + dx^3$

EX :



$q = 0$

$EJ\, v^{IV}(x) = 0$ + BOUNDARY CONDITIONS INVOLVING m.

$v''' = C_1$

$v'' = C_1 x + C_2$

$v' = \bar{C}_1 x^2 + C_2 x + C_3$

$v = \bar{C}_1 x^3 + \bar{C}_2 x + C_3 x + C_4 \Rightarrow$

MAXIMUM POWER OF THE EXACT SOLUTION IS 3.

WHEN $q = 0 \Rightarrow$ FEM IS AN EXACT SOLUTION