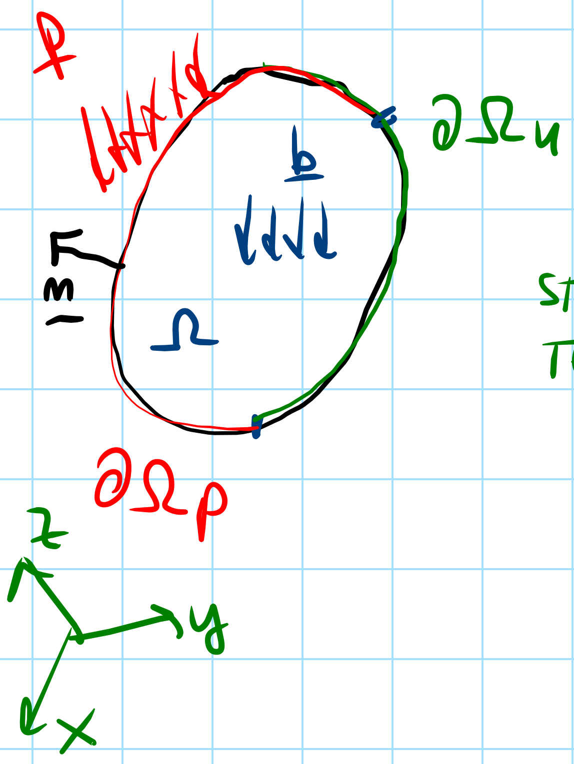
LINEAR ELASTIC PROBLEM

CSM 17/3/26



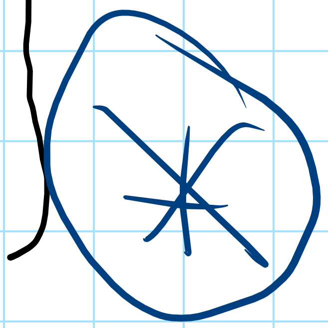
COUCHY STRESS TENSOR

$\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0}$ (3 EQS)

STRAIN TENSOR $\underset{\sim}{\varepsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$ (6 EQS) in Ω

DISPL. FIELD

$\underset{\sim}{\sigma} = \mathbb{C}\, \underset{\sim}{\varepsilon}$ (6 EQS)



15 EQS IN

15 UNKNOWNS

6 : COMPON. OF $\underset{\sim}{\sigma}$

6 : " " $\underset{\sim}{\varepsilon}$

3 : " " $\underline{u}$

$\underset{\sim}{\sigma}(x,y,z)$

$\underset{\sim}{\varepsilon}(x,y,z)$ in Ω

$\underline{u}(x,y,z)$

$\underline{u} = \underline{u}^0$ ← GIVEN in $\partial\Omega_u$

$\underset{\sim}{\sigma}\,\underline{n} = \underline{p}$ in $\partial\Omega_p$

1) DISPLACEMENT FORMULATION (NAVIER)

$$\left.\begin{aligned} &\operatorname{div}\left[\tfrac{1}{2}\,\mathbb{C}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)\right] + \underline{b} = \underline{0} && \text{in } \Omega \\ &\underline{u} = \underline{u}^0 && \text{in } \partial\Omega_u \\ &\left(\tfrac{1}{2}\,\mathbb{C}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)\right)\underline{n} = \underline{p} && \text{in } \partial\Omega_p \end{aligned}\right\}$$

3 EQS. IN

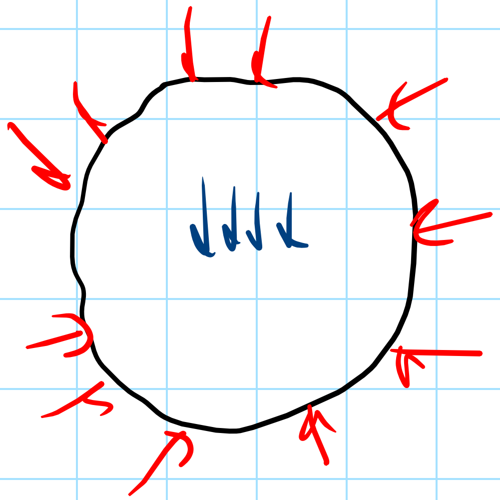
3 UNKNOWNS $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$

IF WE FORMULATE A F.E.M. STARTING FROM STATIONARITY OF T.P.E.

$\Pi(\underline{u})$

2) STRESS FORMULATION (BELTRAMI – MICHELL)

IT IS POSSIBLE TO EXPRESS THE L. EL. PR. PRIMARILY IN TERMS OF THE 6 COMPONENTS OF STRESS $\underset{\sim}{\sigma}$. THIS FORMULATION IS NOT PARTICULARLY SUITABLE WHEN $\partial\Omega_u$ IS PRESENT. WHEN, INSTEAD, $\partial\Omega_u = \emptyset$, THIS APPROACH CAN BE EXPLOITED



$\partial\Omega = \partial\Omega_p$

THE CORRESPONDING F.E.M. IS THAT BASED ON THE COMPLEMENTARY POTENTIAL ENERGY : $\Pi^c(\underset{\sim}{\sigma})$

$$\Pi^c(\underset{\sim}{\sigma}) = \frac{1}{2}\int_{\Omega} \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon}\, dV - \int_{\partial\Omega_p} \underline{p} \cdot \underline{u}\, dS - \int_{\Omega} \underline{b} \cdot \underline{u}\, dV - \int_{\partial\Omega_u} \underbrace{\underline{r}}_{\text{REACTIONS}} \cdot \underline{u}^0\, dS$$

$$= \frac{1}{2}\int_{\Omega} \underset{\sim}{\sigma} \cdot \underset{\sim}{\mathbb{C}}^{-1} \underset{\sim}{\sigma}\, dV - \int \underset{\sim}{\sigma}\,\underline{n} \cdot \underline{u}\, dS \;\ldots\ldots$$

3) MIXED FORMULATION

THE UNKNOWNS ARE $\underline{u}$ (3 COMPONENTS) AND $\underset{\sim}{\sigma}$ (6 COMPONENTS).

THIS IS VERY USEFUL FOR MIXED BOUNDARY CONDITIONS $(\partial\Omega_u, \partial\Omega_p \neq \phi)$

$\Rightarrow$ THE F.E.M. ASSOCIATED IS BASED ON HELLINGER-REISSNER FUNCTIONAL

4) COMPLETE FORMULATION

ALL 15 UNKNOWNS ARE PRIMARY VARIABLES.

THE FEM IS BASED ON HU-WASHIZU FUNCTIONAL

"STRONG" AND WEAK FORMULATIONS IN LINEAR ELASTICITY

THE PROBLEM SET IN $\circledast$ IS THE "STRONG" FORMULATION OF L-EL-PROBLEM

"WEAK" FORMULATIONS ARE THOSE BASED ON INTEGRAL EQUATIONS $\Rightarrow$

$\Rightarrow$ FUNCTIONALS SUITABLY DEFINED:

EX! STAT. OF T.P.E. : $\Pi(\underline{u})$

" " COMPL. T.P.E : $\Pi^c(\underset{\sim}{\sigma})$

" " H.-R. : $HR(\underline{u}, \underset{\sim}{\sigma})$

" " H-W. : $HW(\underline{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma})$

PRINCIPLE OF VIRTUAL WORK

GALERKIN METHODS

HOW TO WRITE THE RIGHT FUNCTIONAL ?

START FORM: $TPE = \underbrace{\frac{1}{2}\int_\Omega \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon}\, dV}_{\text{STRAIN ENERGY IN THE BODY}} - \int_\Omega \underline{b} \cdot \underline{u}\, dV - \int_{\partial\Omega_p} \underline{p} \cdot \underline{u}\, dS - \int_{\partial\Omega_u} \underline{r} \cdot \underline{u}^o\, dS$

$$\Pi(\underline{u}) = \frac{1}{2}\int_\Omega \overbrace{\frac{1}{2}\mathbb{C}(\nabla\underline{u}+\nabla\underline{u}^T)}^{\underset{\sim}{\sigma}} \cdot \overbrace{\frac{1}{2}(\nabla\underline{u}+\nabla\underline{u}^T)}^{\underset{\sim}{\varepsilon}} - \text{-------}$$

$$HW(\underline{u},\underset{\sim}{\varepsilon},\underset{\sim}{\sigma}) = \frac{1}{2}\int \mathbb{C}\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\varepsilon}\, dV - \int \underline{b}\cdot\underline{u}\, dV - \int \underline{p}\cdot\underline{u}\, dS - \int_{\partial\Omega_u} \underset{\sim}{\sigma}\underline{n}(\underline{u}-\underline{u}^o)\, dS + \int_\Omega \underset{\sim}{\sigma}\cdot\overbrace{\left[\frac{1}{2}(\nabla\underline{u}+\nabla\underline{u}^T)-\underset{\sim}{\varepsilon}\right]}^{\text{FORMALLY } \underline{0}} dV$$

$$HR(\underline{u},\underset{\sim}{\sigma}) = \overbrace{\frac{1}{2}\int_\Omega \underset{\sim}{\sigma}\cdot(\nabla\underline{u}+\nabla\underline{u}^T)\, dV - \frac{1}{2}\int_\Omega \underset{\sim}{\sigma}\cdot\mathbb{C}^{-1}\underset{\sim}{\sigma}\, dV}^{\text{STRAIN ENERGY}} - \int_\Omega \underline{b}\cdot\underline{u}\, dV - \int_{\partial\Omega_p} \underline{p}\cdot\underline{u}\, dS - \int_{\partial\Omega_u} \underset{\sim}{\sigma}\underline{n}\cdot(\underline{u}-\underline{u}^o)\, dS$$

RECALL THAT IMPOSING STATIONARITY OF THOSE FUNCTIONALS MEAN

$\Pi(\underline{u}) \longrightarrow \delta\Pi(\underline{u}) = 0$

$HW(\underline{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}) \longrightarrow \delta_{\underline{u}} HW = 0 \;,\; \delta_{\underset{\sim}{\varepsilon}} HW = 0 \;,\; \delta_{\underset{\sim}{\sigma}} HW = 0$

$HR(\underline{u}, \underset{\sim}{\sigma}) \longrightarrow \delta_{\underline{u}} HR = 0 \;,\; \delta_{\underset{\sim}{\sigma}} HR = 0$

FORMAL F.E.M. OF A LINEAR ELASTIC PROBLEM USING STAT. OF $\Pi(\underline{u})$

$\nearrow$ DISPL. FIELD IN THE ELEMENT

(3) $\underline{u}(x,y,z) = \underset{\sim}{N}\,\underline{U}_e$

SHAPE FUNCTION $\leftarrow$ $(3\times 3m)$

$$\underset{\sim}{N} = \left[\begin{array}{ccc|ccc|c|ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_m & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_m & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_m \end{array}\right]$$

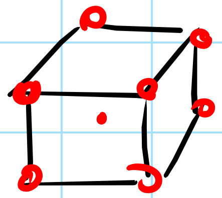
$$u_x(x,y,z) = N_1 u_{1x} + N_2 u_{2x} + \dots + N_m u_{mx}$$

OUR GOAL IS NOW TO OBTAIN:

(6) $\underline{\varepsilon}(x,y,z) = \underset{\sim}{B}\,\underline{U}_e$ $(6\times 3m)$ $(3m)$

NOTE FROM DISPL $\rightarrow$ STRAIN THERE IS DIFFERENTIATION

ELEMENT: HAS m NODES



$\rightarrow \underline{u} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$

N_1: SHAPE FUNCTION

$= 1$ IN NODE 1

$= 0$ IN THE OTHER NODES

$$\underline{U}_e = \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ \vdots \\ u_{mx} \\ u_{my} \\ u_{mz} \end{bmatrix} \quad (3m)$$

$$\underset{\sim}{B} = [\partial]\,\underline{N}$$

$(6\times 3m)$ (6×3) $(3\times 3m)$

$$[\partial] = \begin{bmatrix} \bullet_{,x} & 0 & 0 \\ 0 & \bullet_{,y} & 0 \\ 0 & 0 & \bullet_{,z} \\ 0 & \frac{1}{2}\bullet_{,z} & \frac{1}{2}\bullet_{,y} \\ \frac{1}{2}\bullet_{,z} & 0 & \frac{1}{2}\bullet_{,x} \\ \frac{1}{2}\bullet_{,y} & \frac{1}{2}\bullet_{,x} & 0 \end{bmatrix}$$

$$\left(\underset{\sim}{\varepsilon} = \begin{bmatrix} u_{x,x} & \frac{1}{2}(u_{x,y}+u_{y,x}) & \cdots \\ \vdots & u_{y,y} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \right)$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix}_{(6)}$$

. AFTER $\underline{\varepsilon} = \underset{\sim}{B}\, \underline{U}_e \longrightarrow$ CONST. RELATIONSHIP

$$\underset{(6)}{\underline{\sigma}} = \underset{(6\times6)}{\underset{\sim}{C}}\ \underset{(6)}{\underline{\varepsilon}}$$

VOIGT NOTATION

AT THE ELEMENT LEVEL

$$\Pi(\underline{u}) = \frac{1}{2}\int_{\Omega_e} \underline{\varepsilon}\cdot C\,\underline{\varepsilon}\, dV - \int_{\Omega_e} \underline{b}\cdot\underline{u}\, dV = \frac{1}{2}\int_{\Omega_e} \underset{\sim}{B}\,\underline{U}_e \cdot \underset{\sim}{C}\,\underset{\sim}{B}\,\underline{U}_e\, dV - \int_{\Omega_e} \underline{b}\cdot\underset{\sim}{N}\,\underline{U}_e\, dV$$

$$= \frac{1}{2}\int_{\Omega_e} \underline{U}_e\cdot\underset{\sim}{B}^T\underset{\sim}{C}\,\underset{\sim}{B}\,\underline{U}_e\, dV - \int_{\Omega_e} \underset{\sim}{N}^T b\cdot\underline{U}_e\, dV$$

$$= \underline{U}_e\cdot\underbrace{\left[\frac{1}{2}\int_{\Omega_e}\underset{\sim}{B}^T\underset{\sim}{C}\,\underset{\sim}{B}\, dV\right]}_{\text{STIFFNESS MATRIX OF ELEMENT } \underset{\sim}{K}_e}\underline{U}_e - \underline{U}_e\cdot\underbrace{\left[\int_{\Omega_e}\underset{\sim}{N}^T\underline{b}\, dV\right]}_{\text{VECTOR OF NODAL FORCES } \underline{F}_e}$$

$$\Pi = \frac{1}{2}\,\underline{U}_e\cdot\underset{\sim}{K}_e\,\underline{U}_e - \underline{U}_e\cdot\underline{F}_e$$

$$\frac{\partial\Pi}{\partial\underline{U}_e} = 0 \Rightarrow \boxed{\underset{\sim}{K}_e\,\underline{U}_e - \underline{F}_e = \underline{0}}$$

CHECK DIMENSIONS OF $\underset{\sim}{K}_e \sim \underset{\sim}{B}^T \underset{\sim}{C} \underset{\sim}{B}$

$(3m \times 6)\ (6 \times 6)\ (6 \times 3m)$

$(3m \times 3m)$ OK!

$\underline{F}_e \sim \underset{\sim}{N}^T \underline{b}$

$(3m \times 3)\ (3)$

$(3m)$