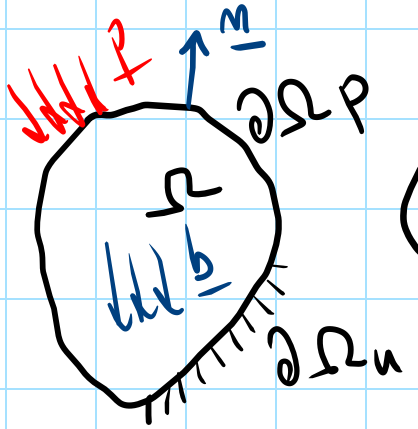
LINEAR ELASTIC PROBLEM

21/3/25



$$\Omega \begin{cases} \operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0} & (3) \\ \underset{\sim}{\varepsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) & (6) \\ \underset{\sim}{\sigma} = \mathbb{C}\, \underset{\sim}{\varepsilon} & (6) \end{cases}$$

(15)

B. CONDITIONS

$\underset{\sim}{\sigma}\, \underline{n} = \underline{p}$ on $\partial\Omega_p$

$\underline{u} = \underline{\bar{u}}$ on $\partial\Omega_u$ ($\underline{\bar{u}}$: ASSIGNED DISPL on $\partial\Omega_u$)

UNKNOWNS OF THE PROBLEM

$\underline{u}$: (3)

$\underset{\sim}{\varepsilon}$: (6)

$\underset{\sim}{\sigma}$: (6)

(15)

MATH. ANAL HAS SHOWN THAT, UNDER REGULAR HYPOTHESES, THE PROBLEM HAS A SOLUTION (EXISTENCE); MOREOVER, WITH A SIMPLE ARGUMENT, IT CAN BE SHOWN THAT IT IS UNIQUE (WITH $\mathbb{C}$ POSITIVE DEFINITE - KIRCHHOFF THEOREM)

MOREOVER, EFFECT SUPERPOSITION IS VALID.

DISPLACEMENTS METHOD (FOCUS ON $\underline{u}$)

$$\operatorname{div}\left[\frac{1}{2}\mathbb{C}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)\right] + \underline{b} = \underline{0} \quad (\Omega)$$

(3 EQS IN 3 UNKOWNS)

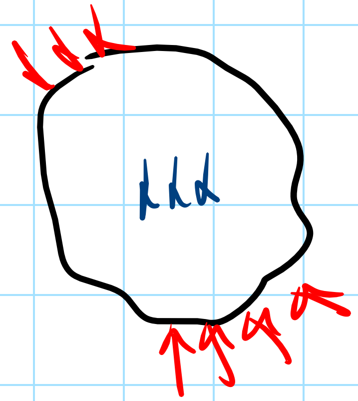
$$\left.\begin{aligned} \frac{1}{2}\mathbb{C}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)\underline{n} &= \underline{p} \\ \underline{u} &= \underline{\bar{u}} \end{aligned}\right] \text{b.c.'s}$$

NAVIER FORMULATION

STRESS METHOD (BELTRAMI - MICHELL FORMULATION)

$\underset{\sim}{\sigma}$ AS UNKNOWN (6 COMPONENTS) ; NOT VERY USEFUL WHEN $\partial\Omega_u \neq \phi$

USEFUL WHEN $\partial\Omega_u = \phi$:



IN GENERAL, THE SOLUTION OF L.E.P. IS A 'MIXED' PROBLEM WHERE UNKNOWNS OF DIFFERENT TYPE ARE IN.

WEAK FORMULATIONS OF L.E.P.

STATIONARITY OF TOTAL POTENTIAL ENERGY (DISPLACEMENT METHOD)

COMPLEMENTARY T.P.E. (STRESS METHOD)

FUNCTIONAL OF HELLINGER-REISSNER ($\underline{u}$, $\underset{\sim}{\sigma}$ UNKNOWNS : 9 COMPONENTS)

" " HU-WASHIZU (ALL 15 UNKNOWNS)

H.-R. FUNCTIONAL

$$HR(\underline{u},\underset{\sim}{\sigma}) = \underbrace{\frac{1}{2}\int_\Omega \underset{\sim}{\sigma}\cdot(\nabla\underline{u}+\nabla\underline{u}^T)\,dV - \frac{1}{2}\int_\Omega \underset{\sim}{\sigma}\cdot\mathbb{C}^{-1}\underset{\sim}{\sigma}\,dV} - \int_{\partial\Omega_p} \underline{p}\cdot\underline{u}\,dS - \int_\Omega \underline{b}\cdot\underline{u}\,dV - \int_{\partial\Omega_u} \underset{\sim}{\sigma}\underline{n}\cdot(\underline{u}-\underline{\bar{u}})\,dS$$

$$\frac{1}{2}\int\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} = \int\sigma\cdot\varepsilon - \frac{1}{2}\int\sigma\cdot\varepsilon = \frac{1}{2}\int\underset{\sim}{\sigma}\cdot(\nabla u+\nabla u^T)\,dV - \frac{1}{2}\int\underset{\sim}{\sigma}\cdot\mathbb{C}^{-1}\underset{\sim}{\sigma}$$

$$\Rightarrow \boxed{\delta_u HR = 0 \quad ; \quad \delta_\sigma HR = 0}$$ STATIONARITY OF HR

HU-WASHIZU FUNCTIONAL (FREE FUNCTIONAL)

$$HW(\underline{u},\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}) = \frac{1}{2}\int_\Omega \mathbb{C}\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\varepsilon}\,dV - \int_{\partial\Omega_p}\underline{p}\cdot\underline{u}\,dS - \int_\Omega \underline{b}\cdot\underline{u}\,dV + \int_\Omega \underset{\sim}{\sigma}\cdot\left[\frac{1}{2}(\nabla\underline{u}+\nabla\underline{u}^T) - \underset{\sim}{\varepsilon}\right]dV$$
$$- \int_{\partial\Omega_u}\underset{\sim}{\sigma}\underline{n}\cdot(\underline{u}-\underline{\bar{u}})\,dS$$

$$\boxed{\begin{aligned}&\delta_u HW = 0\\ &\delta_\sigma HW = 0\\ &\delta_\varepsilon HW = 0\end{aligned}}$$ STATIONARITY OF HW

STATIONARITY OF T. P. E $(\Pi(\underline{u}))$

$$\Pi(\underline{u}) = \frac{1}{2}\int_{\Omega} \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon}\, dV - \int_{\partial\Omega_p} \underline{p} \cdot \underline{u}\, dS - \int_{\Omega} \underline{b} \cdot \underline{u}\, dV$$

$$\Pi(\underline{u}) = \frac{1}{2}\int \frac{1}{2}\mathbb{C}(\nabla\underline{u}+\nabla\underline{u}^T) \cdot \frac{1}{2}(\nabla\underline{u}+\nabla\underline{u}^T)\, dV - \ldots$$

PRINCIPLE OF STATIONARITY OF T. P. E. $(\delta\Pi^{(1)}(\underline{u})=0,\ \forall\delta\underline{u})$

WITHIN THE SET OF DISPL. FIELDS $\underline{u}$ KINEMATICALLY ADMISSIBLE*, THE ONE THAT SOLVE THE L. E. P. CORRESPONDS TO THE CONDITION

$$\boxed{\delta\Pi^{(1)}(\underline{u})=0,\ \forall\delta\underline{u}} \qquad (\Leftrightarrow \text{EQUILIBRIUM})$$

SKETCH OF PROOF :

$$\Pi(\underline{u}+\delta\underline{u}) = \Pi(\underline{u})+\delta\Pi = \frac{1}{2}\int \mathbb{C}\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\varepsilon} - \int\underline{b}\cdot\underline{u} - \int\underline{p}\cdot\underline{u} + \overbrace{\int\mathbb{C}\varepsilon\cdot\delta\varepsilon - \int\underline{b}\cdot\delta u - \int p\cdot\delta u}^{\delta\Pi^{(1)}} + \underbrace{\frac{1}{2}\int\mathbb{C}\delta\varepsilon\cdot\delta\varepsilon}_{\delta\Pi^{(2)}} > 0 \Leftrightarrow$$

$\mathbb{C}$ IS POSITIVE DEFINITE

$\delta\Pi^{(1)}$ IS AN EXPRESSION OF THE THEOREM OF VIRTUAL WORK, AND BY IMPOSING $\delta\Pi^{(1)}=0$ WE CAN SHOW THAT $\{\underset{\sim}{\sigma}, \underline{p}, \underline{b}\}$ IS IN EQUILIBRIUM

*: $\underline{u}(P) = \bar{\underline{u}}$ on $\partial\Omega_u$

DISCRETAZION OF THE L.E.P. BASED ON STATIONARITY OF T.P.E. (BASED ON DISPL. METHOD)

FOR AN ELEMENT (e)

$$\tilde{\underline{u}}_e(x,y,z) = \begin{bmatrix} \sum_{K=1}^{m} \Phi_K(x,y,z)\, u^e_{Kx} \\ u^e_{Ky} \\ u^e_{Kz} \end{bmatrix} = \underset{(3\times 3m)}{\underset{\sim}{N}}\ \underset{(3m)}{\underline{D}_e}$$

APPROX DISPL FIELD (3)

$$\underline{D}_e = \begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ \vdots \\ u^e_{mz} \end{bmatrix} \ (3m)$$

VECTOR OF NODAL DISPLACEMENTS

Φ_i : SHAPE FUNCTIONS

$$\underset{\sim}{N} = \begin{bmatrix} \Phi_1 & 0 & 0 & \Phi_2 & 0 & 0 & \cdots \\ 0 & \Phi_1 & 0 & 0 & \Phi_2 & 0 & \cdots \\ 0 & 0 & \Phi_1 & 0 & 0 & \Phi_2 & \cdots \end{bmatrix}$$

(3x3m)

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} u_{x,x} & \frac{1}{2}(u_{x,y}+u_{y,x}) & \\ & u_{y,y} & \\ & & \end{bmatrix}$$

$$\underline{\varepsilon} = \underset{(6\times 3)}{[\partial]}\, \underline{u}$$

COMPATIBILITY MATRIX (?)

NOW WE HAVE A GOAL : $\underset{(6)}{\tilde{\underline{\varepsilon}}(x,y,z)} = \underset{(6\times 3m)}{\underset{\sim}{B}}\ \underset{(3m)}{\underline{D}_e}$

$$[\partial] = \begin{bmatrix} \cdot_{,x} & 0 & 0 \\ 0 & \cdot_{,y} & 0 \\ 0 & 0 & \cdot_{,z} \\ 0 & \frac{1}{2}\cdot_{,z} & \frac{1}{2}\cdot_{,y} \\ \frac{1}{2}\cdot_{,z} & 0 & \frac{1}{2}\cdot_{,x} \\ \frac{1}{2}\cdot_{,y} & \frac{1}{2}\cdot_{,x} & 0 \end{bmatrix}$$

$$\underset{\sim}{B} = \underset{(6\times 3)}{[\partial]}\ \underset{(3\times 3m)}{\underset{\sim}{N}}$$

OK!

$$\underset{(6)}{\tilde{\underline{\sigma}}(x,y,z)} = \underset{(6\times 6)}{C}\ \underset{(6)}{\tilde{\underline{\varepsilon}}}$$

(USING VOIGT NOTATION)