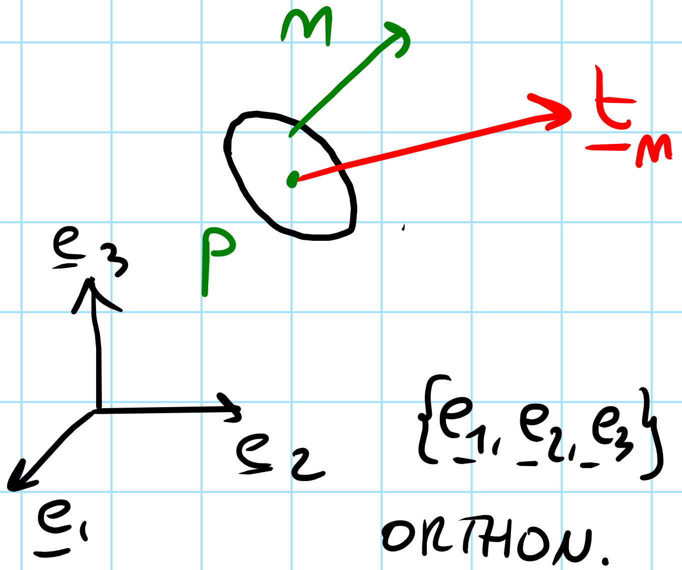
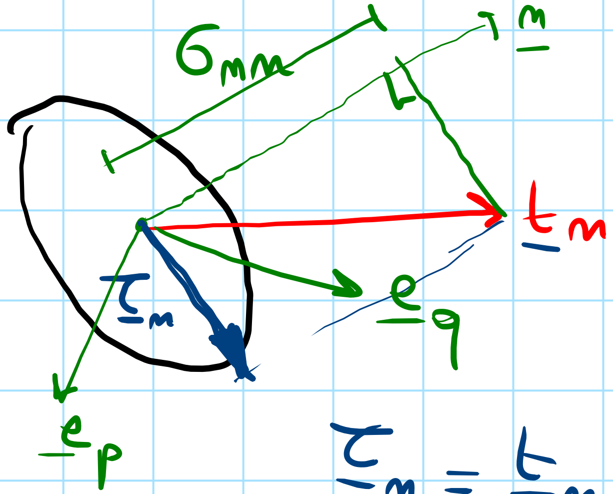
CARTESIAN COMPONENTS OF THE TRACTION VECTOR



$$t_{n_i} = \underline{t}_n \cdot \underline{e}_i \qquad \underline{t}_n = \begin{bmatrix} t_{n_1} \\ t_{n_2} \\ t_{n_3} \end{bmatrix}$$

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ ORTHON. BASIS

SPECIAL COMPONENTS OF THE TRACTION VECTOR



$\sigma_{nn} = \underline{t}_n \cdot \underline{n}$: NORMAL STRESS
- > 0 : TENSION
- < 0 : COMPRESSION

$\underline{\tau}_n$: SHEAR STRESS VECTOR

$$\underline{\tau}_n = \underline{t}_n - \sigma_{nn}\,\underline{n}$$

$$\left.\begin{aligned} \tau_{np} &= \underline{\tau}_n \cdot \underline{e}_p \\ \tau_{nq} &= \underline{\tau}_n \cdot \underline{e}_q \end{aligned}\right\} \text{SHEAR STRESS COMPONENTS}$$

$$\left.\begin{aligned} &\sigma_{nn} \\ &\tau_{np} \\ &\tau_{nq} \end{aligned}\right] \text{SPECIAL COMPONENTS}$$

$\underline{e}_p \perp \underline{e}_q$: BASIS IN THE PLANE

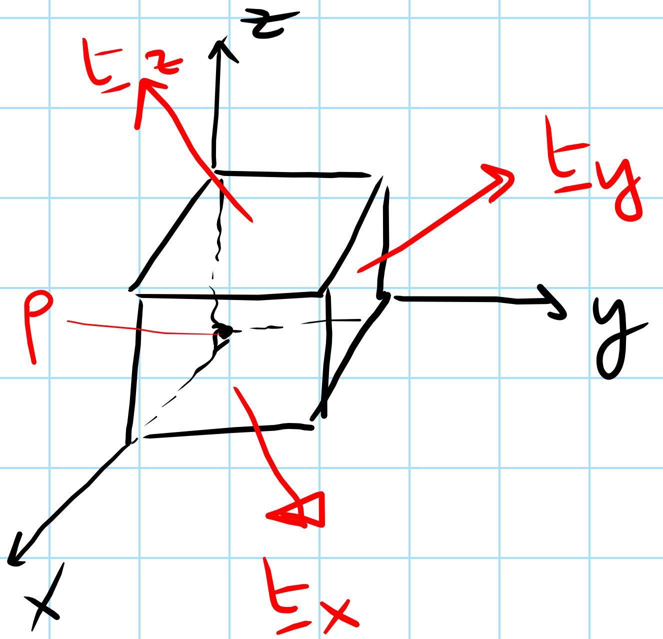
CAUCHY'S THEOREM AND STRESS TENSOR

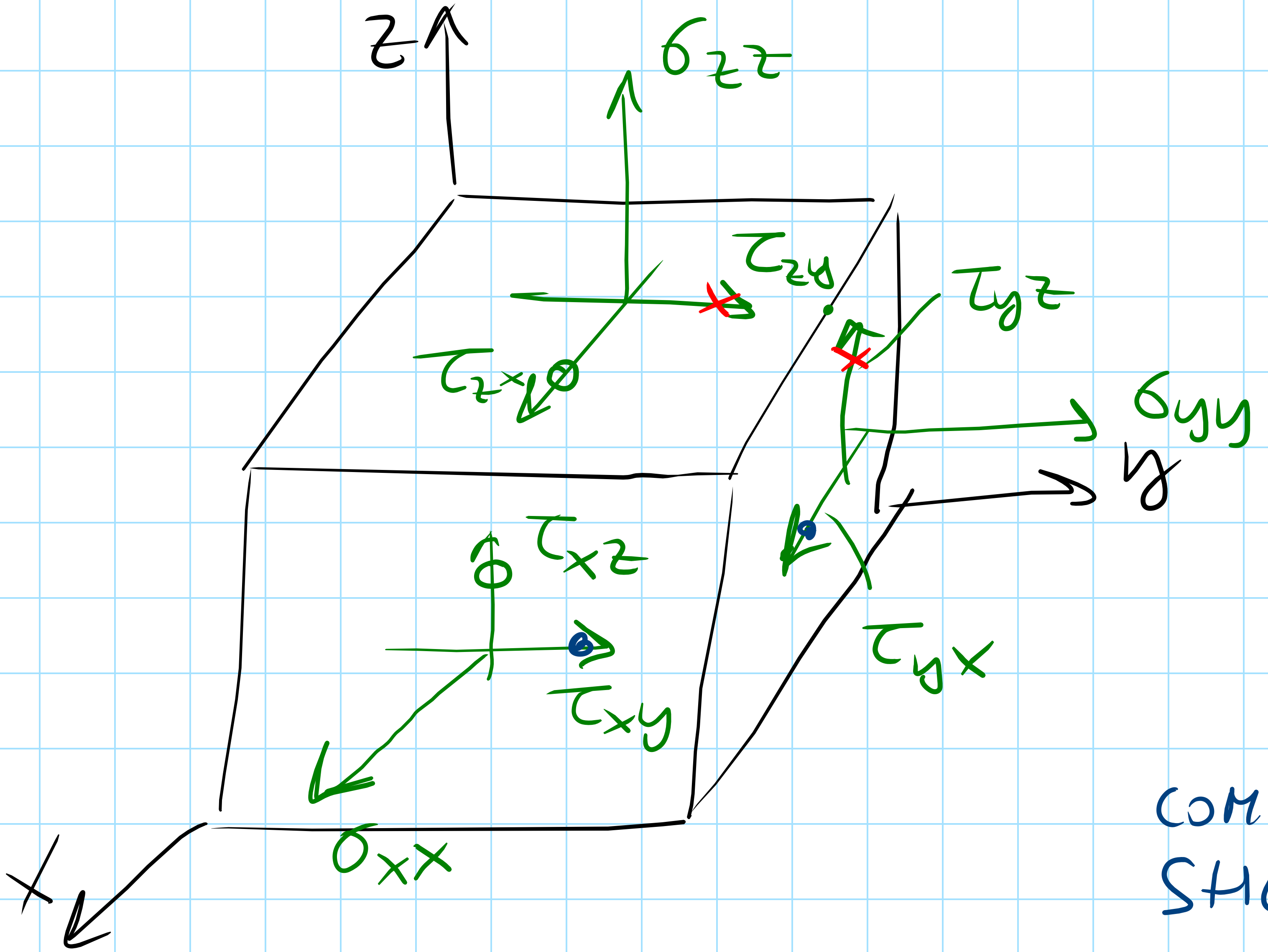
WHAT DOES IT MEAN "TO KNOW THE STRESS AT A POINT"?

$P, \underline{n} \longrightarrow \underline{t}(P, \underline{n})$

$P, \underline{m} \longrightarrow \underline{t}(P, \underline{m})$ $\Big\} \not\exists$

IT MEANS TO KNOW ALL STRESS VECTORS REFERRED TO GENERIC $\underline{n}$ $\longrightarrow$ INFINITE INFORMATION





WE CAN SHOW THAT DUE TO BALANCE OF MOMENTS

$$\tau_{xy} = \tau_{yx}$$

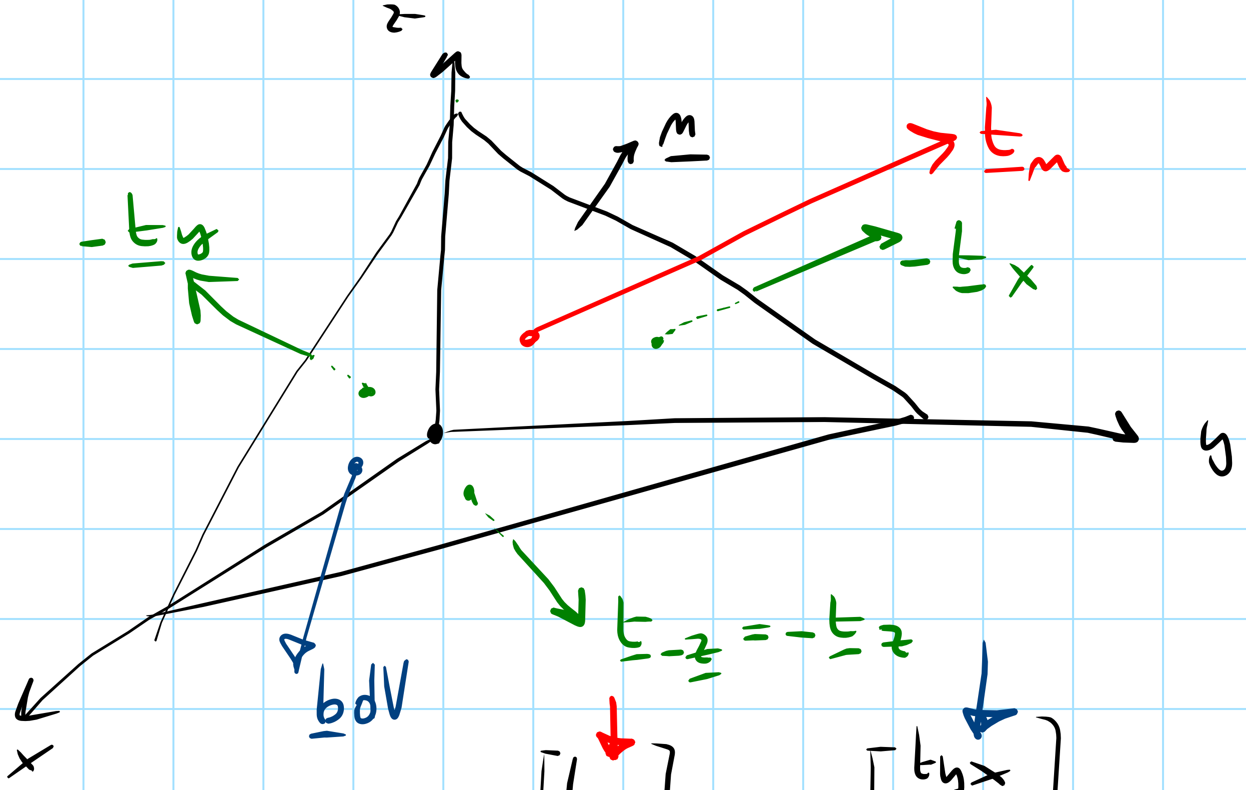
$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

COMPLEMENTARITY OF SHEAR STRESSES

CAUCHY'S TH. (TETRAHEDRON THEOREM)

IN $P \in \Omega$, $\underline{t}(P, \underline{m})$ $\forall \underline{m}$ ARE KNOWN IF THE TRACTION VECTORS RELATED TO 3 ORTHOGONAL PLANES ARE KNOWN (IN THE NEIGHBORHOOD OF P)



WITH BALANCE OF FORCES:

$$\underline{t}_m = \underline{t}_x m_x + \underline{t}_y m_y + \underline{t}_z m_z$$

$$\underline{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

$$\left[\underline{t}_m\right] = \begin{bmatrix} t_{xx} \\ t_{xy} \\ t_{xz} \end{bmatrix} m_x + \begin{bmatrix} t_{yx} \\ t_{yy} \\ t_{yz} \end{bmatrix} m_y + \begin{bmatrix} t_{zx} \\ t_{zy} \\ t_{zz} \end{bmatrix} m_z = \begin{bmatrix} \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \left[\underset{\sim}{\sigma}\right]\left[\underline{m}\right] = \underset{\sim}{\sigma}\,\underline{m}$$

CAUCHY STRESS TENSOR

COMPONENTS OF $\underset{\sim}{\sigma}$

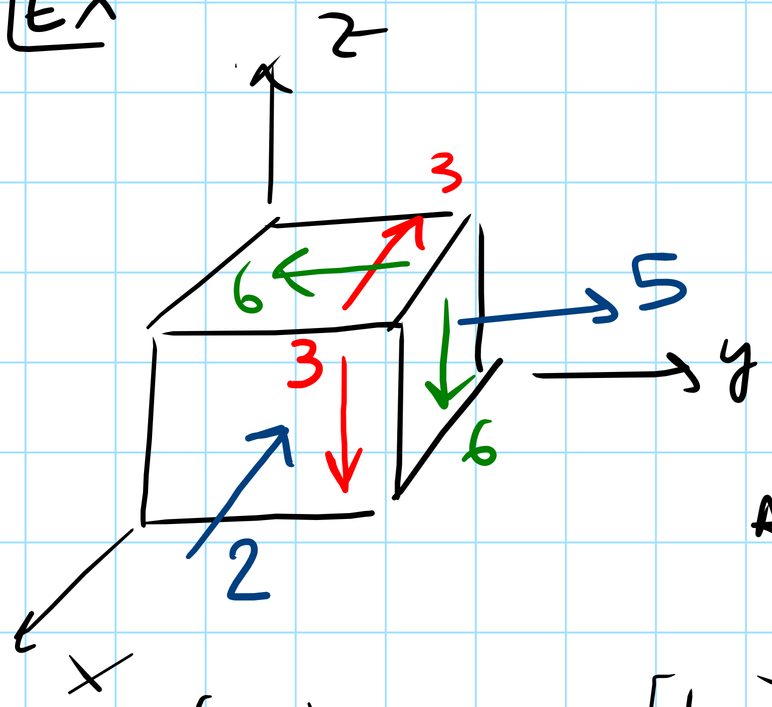
SHEAR STRESSES ARE COMPLEMENTARY

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \Rightarrow \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$$

$\underset{\sim}{\sigma}$ IS SYMMETRIC

6 INDEPENDENT VALUES

EX



(MPa)

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} -2 & 0 & -3 \\ 0 & +5 & -6 \\ -3 & -6 & 0 \end{bmatrix} \quad (MPa)$$

AFTER $[\underset{\sim}{\sigma}]$, $\underline{t}_q$ WITH $\underline{q} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$[\underline{t}_q] = \begin{bmatrix} -2 & 0 & -3 \\ 0 & 5 & -6 \\ -3 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -5/\sqrt{3} \\ -1/\sqrt{3} \\ -9/\sqrt{3} \end{bmatrix}$$

(CARTESIAN COMPONENTS OF $\underline{t}_q$)

$$\sigma_{qq} = \underline{t}_q \cdot \underline{q} = \frac{1}{3}\left(-5 \cdot 1 + (-1) \cdot 1 + (-9) \cdot 1\right) = -5 \text{ MPa}$$

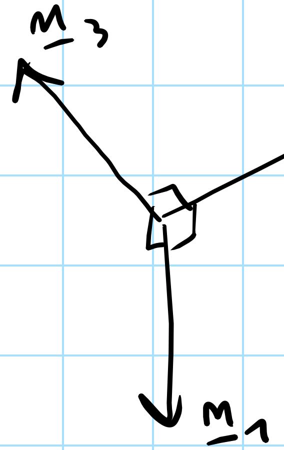
EIGENVALUES AND EIGENVECTORS OF $\underset{\sim}{\sigma}$

σ_i IS AN EIGENVL AND $\underline{n}_i$ IS THE ASSOCIATED EIGENVCT IF: $\underset{\sim}{\sigma}\,\underline{n}_i = \sigma_i\,\underline{n}_i$

PHYSICALLY, THE THREE σ_i $(i=1,2,3)$ ARE CALLED PRINCIPAL STRESSES

$\underline{n}_i$ " " PRINCIPAL DIRECTIONS OF STRESS

AS $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \Rightarrow \sigma_i$ $(i=1,2,3)$ ARE $\mathbb{R}$ AND $\underline{n}_i$ ARE MUTUALLY ORTHOGONAL!

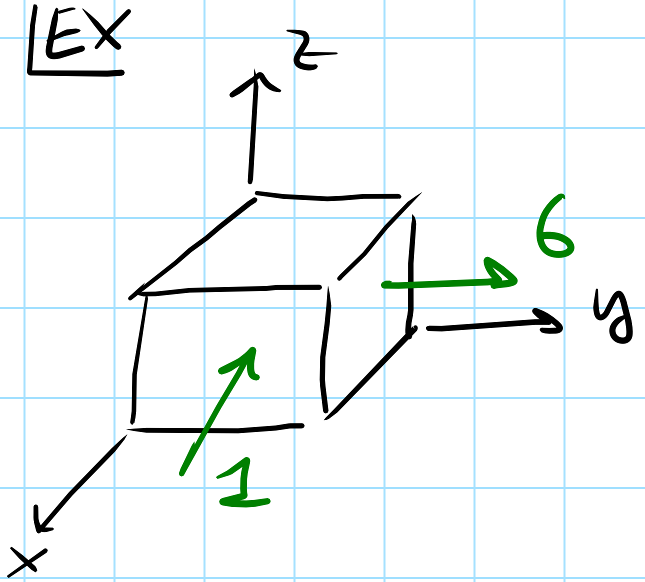


$$\det(\underset{\sim}{\sigma} - \sigma_i \underset{\sim}{I}) = 0$$

$\underset{\sim}{\sigma}$ IS DIAGONAL → $\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix}$ in the BASIS $\{\underline{n}_1, \underline{n}_2, \underline{n}_3\}$

PRINCIPAL STRESSES CAN HELP A LOT IN CLASSIFYING THE TYPES OF STRESS STATE

EX



$$[\underset{\sim}{\sigma}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = 6$

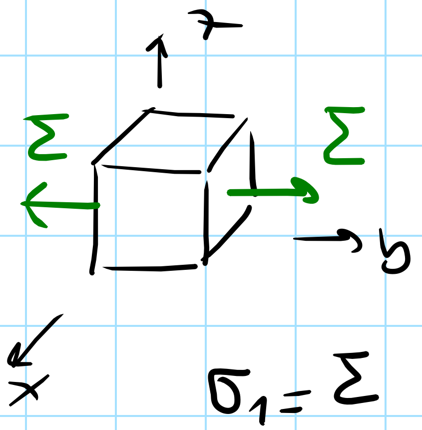
$\sigma_2 = 0$

$\sigma_3 = -1$

X, Y, Z ARE ALREADY A PRINCIPAL SYSTEM

RELEVANT STRESS STATES

- UNIAXIAL STRESS STATE



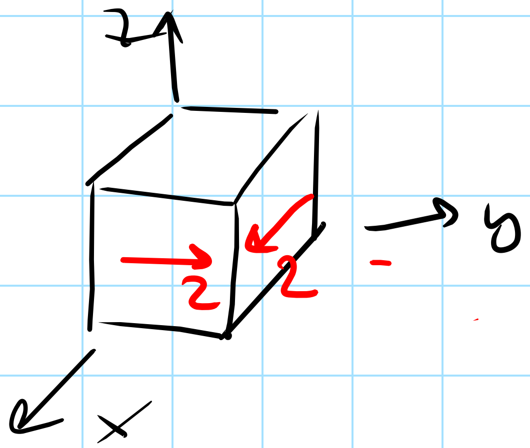
$$[\underset{\sim}{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

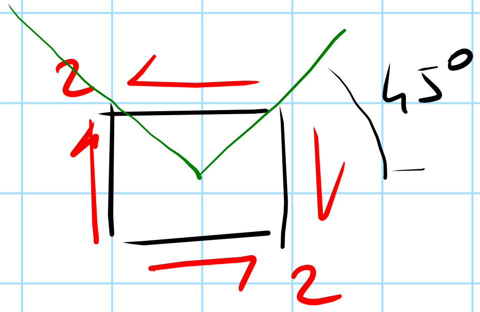
$\sigma_1 = \Sigma$

$\sigma_2 = \sigma_3 = 0$

2 VANISHING σ_i

- PURE SHEAR





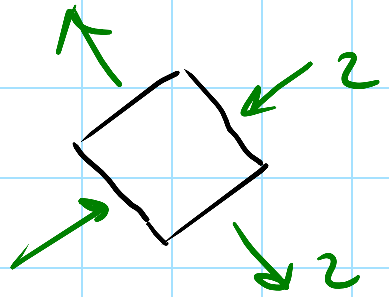
$$[\underset{\sim}{\sigma}] = \begin{bmatrix} 0 & +2 & 0 \\ +2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|\underset{\sim}{\sigma} - \sigma_i \underset{\approx}{I}| = 0$$

→ $\sigma_1 = +2$

→ $\sigma_2 = 0$

→ $\sigma_3 = -2$

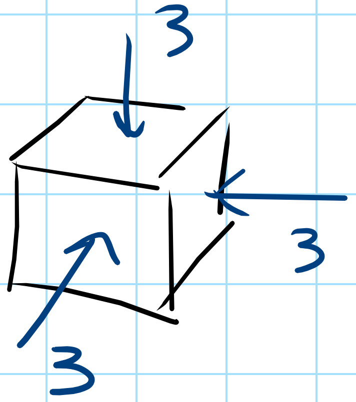


PRINCIPAL DIR: 45°

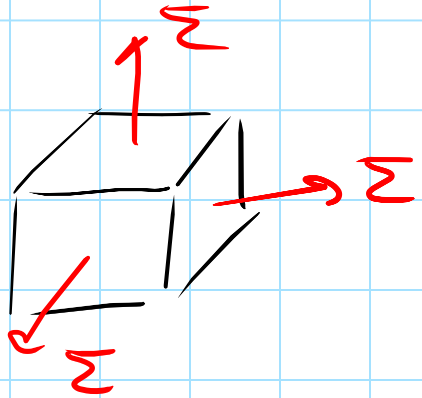
– HYDROSTATIC STRESS STATE

$$\sigma_1 = \sigma_2 = \sigma_3 = \Sigma$$

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & \Sigma \end{bmatrix} \Rightarrow \underset{\sim}{\sigma} = \Sigma \, \underset{\sim}{I}$$

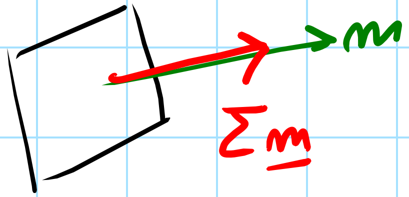


$$[\underset{\sim}{\sigma}] = \begin{bmatrix} -3 & & \\ & -3 & \\ & & -3 \end{bmatrix}$$



GENERIC $\underline{m}$

$$\underline{t}_m \, ? : \; \underset{\sim}{\sigma} \, \underline{m} = \Sigma \, \underset{\sim}{I} \, \underline{m} = \Sigma \, \underline{m}$$



$\underline{m}$ IS A PRINCIPAL DIRECTION ($\forall \underline{m}$)

HYDROSTATIC STRESS AND DEVIATORIC STRESS

$$\underset{\sim}{\sigma} = \sigma_m \underset{\sim}{I} + \underset{\sim}{\sigma}^{DEV} \quad , \quad \sigma_m = \frac{tr\,\underset{\sim}{\sigma}}{3} \; : \text{ HYDROSTATIC PRESSURE } \in \mathbb{R}$$

$$\underset{\sim}{\sigma}^{DEV} = \underset{\sim}{\sigma} - \sigma_m \underset{\sim}{I} \quad ; \quad tr\,\underset{\sim}{\sigma}^{DEV} = 0$$

| PROOF: $tr\,\underset{\sim}{\sigma}^{DEV} = tr\,\underset{\sim}{\sigma} - \sigma_m\, tr\,\underset{\sim}{I} = tr\,\underset{\sim}{\sigma} - 3\sigma_m = 0$

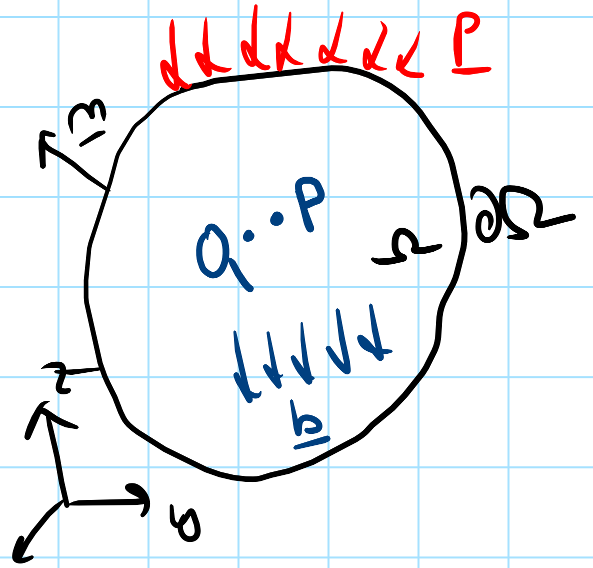
THIS DECOMPOSITION OF $\underset{\sim}{\sigma}$ IS FUNDAMENTAL TO FORMULATE FAILURE CRITERIA FOR MOST OF THE MATERIALS

EX:

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} 3 & 9 & 1 \\ 9 & 5 & 0 \\ 1 & 0 & -10 \end{bmatrix} \Rightarrow \sigma_m = -\frac{2}{3}\,\text{MPa} \; , \quad [\underset{\sim}{\sigma}^{DEV}] = \begin{bmatrix} 3-(-\frac{2}{3}) & 9 & 1 \\ 9 & 5-(-\frac{2}{3}) & 0 \\ 1 & 0 & -10-(-\frac{2}{3}) \end{bmatrix}$$

$$= \begin{bmatrix} 11/3 & 9 & 1 \\ 9 & 17/3 & 0 \\ 1 & 0 & -28/3 \end{bmatrix}$$

BALANCE EQUATIONS IN A CONTINUUM (CONDITIONS THAT MUST BE SATISFIED BY $\underset{\sim}{\sigma}$ CHANGING POINT IN A BODY)



Ω IN EQUILIBRIUM

$P \to \underset{\sim}{\sigma}(P)$
$Q \to \underset{\sim}{\sigma}(Q)$?

HOW $\underset{\sim}{\sigma}(P)$ AND $\underset{\sim}{\sigma}(Q)$ ARE RELATED?

IT CAN BE PROVED THAT COMPONENTS OF $\underset{\sim}{\sigma}$ MUST SATISFY THE FOLLOWING EQUATIONS

$$\begin{cases} \sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} + b_x = 0 \\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} + b_y = 0 \\ \tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} + b_z = 0 \end{cases} \quad \text{in } \Omega$$

$$\boxed{\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0}}$$

on $\partial\Omega$: $\boxed{\underset{\sim}{\sigma}\,\underline{m} = \underline{p}}$ BOUNDARY CONDITIONS

RECALL THAT $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$