A NOTE ON THE RELATIONSHIP BETWEEN $\underline{u}(P)$ AND $\underset{\sim}{\varepsilon}(P)$

CSM, 26/3/26

$\underline{u}(P)$ KNOWN $\longrightarrow$ $\underset{\sim}{\varepsilon} = \frac{1}{2}\,\mathrm{SYM}(\nabla \underline{u})$

3 COMPONENTS — 6 COMPONENTS

IN "SIMULATIONS" WE USUALLY COMPUTE FIRST $\underset{\sim}{\varepsilon}$ AND THEN, IF NEEDED, COMPONENTS OF $\underline{u}(P)$ (THROUGH AN "INTEGRATION")

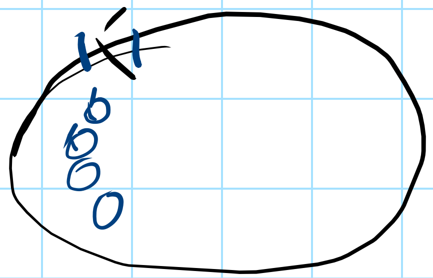
$\underset{\sim}{\varepsilon}$ KNOWN $\longrightarrow$ $\underline{u}(P)$?

MATH HAS SHOWN THAT, IN ORDER TO HAVE $\underline{u}(P)$ "COMPATIBLE" ε_{ij} MUST SATISFY:

$$\boxed{\mathrm{rot}\,\mathrm{rot}\,\underset{\sim}{\varepsilon} = \underset{\sim}{0}}$$

BIANCHI'S IDENTITY

"INTERNAL COMPATIBILITY"

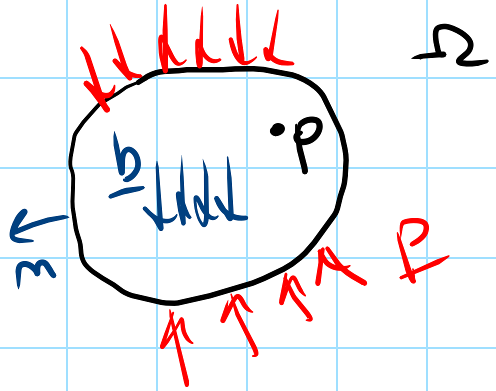


$$\oint A\,dl$$

$$\int_S \mathrm{rot}\,A\,dS$$

VIRTUAL WORK THEOREM (FUNDAMENTAL IDENTITY OF MECHANICS)

- STATICS, EQUILIBRIUM OF A BODY Ω (a)

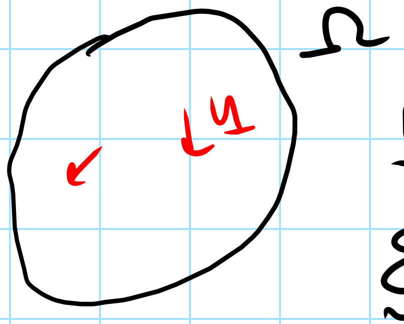


Ω IN EQUIL.

$$\exists\, \underset{\sim}{\sigma}^a(P) : \begin{cases} \operatorname{div} \underset{\sim}{\sigma}^a + \underline{b}^a = \underline{0} & \text{in } \Omega \\ \underset{\sim}{\sigma}^a \underline{m} = \underline{p}^a & \text{on } \partial\Omega \end{cases}$$

$\{\underline{b}^a, \underline{p}^a; \underset{\sim}{\sigma}^a\}$ IS A STATICALLY ADMISSIBLE SYSTEM

- KINEMATICS (b)



$\underline{u}^b(P)$

$$\underset{\sim}{\varepsilon}^b(P) = \frac{1}{2}\left(\nabla \underline{u}^b + \nabla \underline{u}^{bT}\right)$$

$\{\underline{u}^b, \underset{\sim}{\varepsilon}^b\}$ IS A KINEMATICALLY ADMISSIBLE SYSTEM

SYSTEM (a) AND SYSTEM (b) ARE COMPLETELY INDEPENDENT.

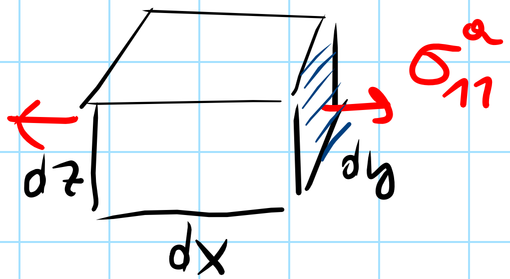
building the two quantities:

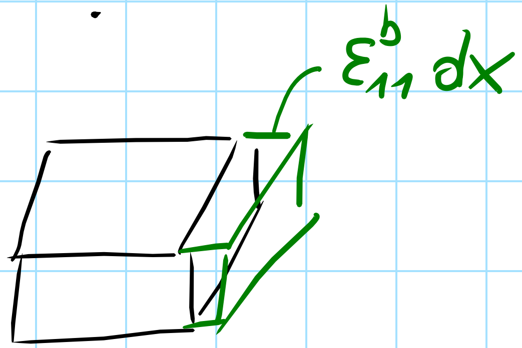
$$L_{v\,ext} = \int_{\Omega} \underline{b}^a \cdot \underline{u}^b \, dV + \int_{\partial\Omega} \underline{p}^a \cdot \underline{u}^b \, dA$$ (EXT VIRTUAL WORK)

$$L_{v\,int} = \int_{\Omega} \underset{\sim}{\sigma}^a \cdot \underset{\sim}{\varepsilon}^b \, dV$$ (INTERNAL VIRTUAL WORK)

WHY $L_{v\,int}$ has dimension of work?

$\sigma_{11}^a \varepsilon_{11}^b \, dV$





$$\overset{dF}{\left(\sigma_{11}^a \, dy\, dz\right)} \cdot \overset{du}{\varepsilon_{11}^b \, dx}$$

$$= \sigma_{11}^a \, \varepsilon_{11}^b \, \underbrace{dx\,dy\,dz}_{dV}$$: LAVORO

THEOREM: GIVEN A BODY Ω, A SET $\{\underline{b}^a, \underline{p}^a, \underset{\sim}{\sigma}^a\}$ STAT ADMISS AND A SET $\{\underline{u}^b, \underset{\sim}{\varepsilon}^b\}$ KINEM ADMISS

THEN THE IDENTITY

$$\boxed{L_{v\,ext} = L_{v\,int}}$$

CAN BE ESTABLISHED

BEFORE THE PROOF, RECALL THE GREEN'S LEMMA:

$$\int_{\partial\Omega} \underset{\sim}{A}\,\underline{m}\cdot\underline{u}\; dS = \int_{\Omega} \operatorname{div}\underset{\sim}{A}\cdot\underline{u} + \underset{\sim}{A}\cdot\nabla\underline{u}\; dV$$

$$\int_{\partial\Omega} A_{ij} m_j u_i\; dS = \int_{\partial\Omega} A_{ij} u_i m_j\; dS = \int_{\Omega} \operatorname{div}\left(A_{ij} u_i\right) dV = \int_{\Omega} \left(A_{ij} u_i\right)_{,j}\; dV$$

$$= \int_{\Omega} A_{ij,j}\, u_i + A_{ij}\, u_{i,j}\; dV = \int_{\Omega} \operatorname{div}\underset{\sim}{A}\cdot\underline{u} + \underset{\sim}{A}\cdot\nabla\underline{u}\; dV$$

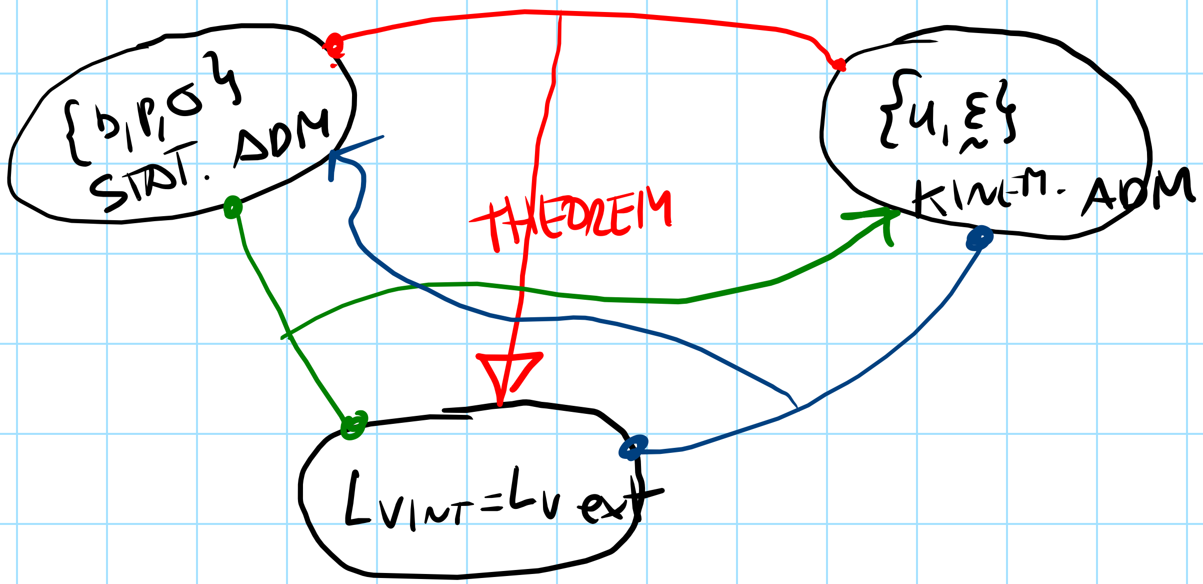
PROOF

$$\boxed{L_{vext}} = \int_{\Omega} \underline{b}^a\cdot u^b\, dV + \int_{\partial\Omega} \underline{p}^a\cdot\underline{u}^b\, dA = \int \underline{b}\cdot\underline{u}\, dV + \int_{\partial\Omega} \underset{\sim}{\sigma}^a\,\underline{m}\cdot\underline{u}^b\, dA = \overbrace{\int_{\Omega} \underline{b}^a\cdot\underline{u}^b\, dV + \int_{\Omega} \operatorname{div}\underset{\sim}{\sigma}^a\cdot\underline{u}^b}^{0} + \int_{\Omega} \underset{\sim}{\sigma}^a\cdot\nabla\underline{u}^b\, dV$$

$$= \int \underset{\sim}{\sigma}^a\cdot\nabla\underline{u}^b\, dV$$; VISTO CHE $\underset{\sim}{\sigma}^a$ è SYM ALLORA

$$= \int \underset{\sim}{\sigma}^a\cdot\underset{\sim}{\varepsilon}^b\, dV = \boxed{L_{v\,int}}$$

CHART OF VIRTUAL WORK PRINCIPLE

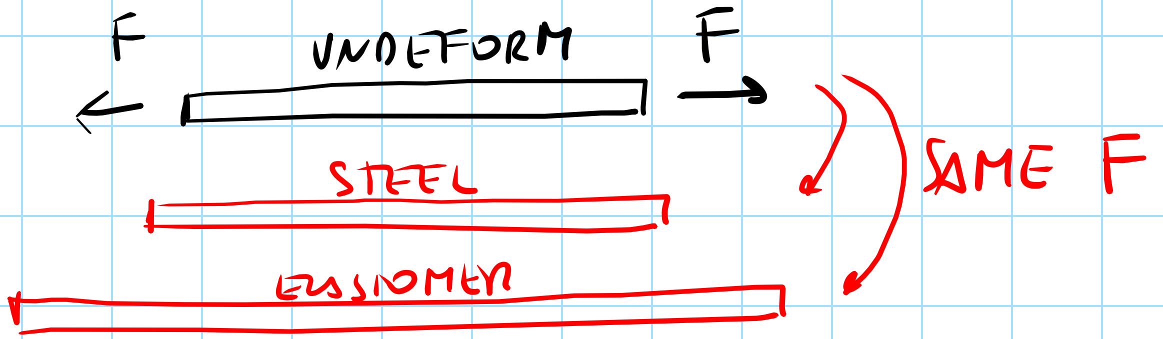


- THIS CHART ARE VALID FOR EACH BEHAVIOUR OF THE MATERIAL OF Ω : NO ASSUMPTIONS REGARDIG CONSTITUTIVE LAW HAVE BEEN PUT FORWARD

- BASIC TOOL TO FORMULATE STRUTURAL MODELS IN ENGINEERING
  USEFUL ALSO IN NUMERICAL SCHEMES FOR SIMULATIONS

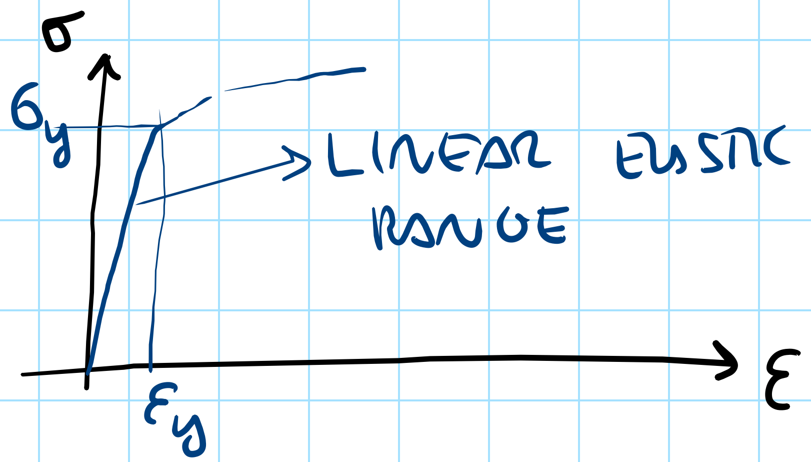
LINEAR ELASTICITY (SPECIAL CASE OF CONSTITUTIVE LAW)

CONSTITUTIVE EQUATION/LAW: RELATIONSHIP BETWEEN $\underset{\sim}{\sigma}$ AND $\underset{\sim}{\varepsilon}$ AT A POINT OF Ω.



INPUT: SAME GEOMETRY AND SAME F

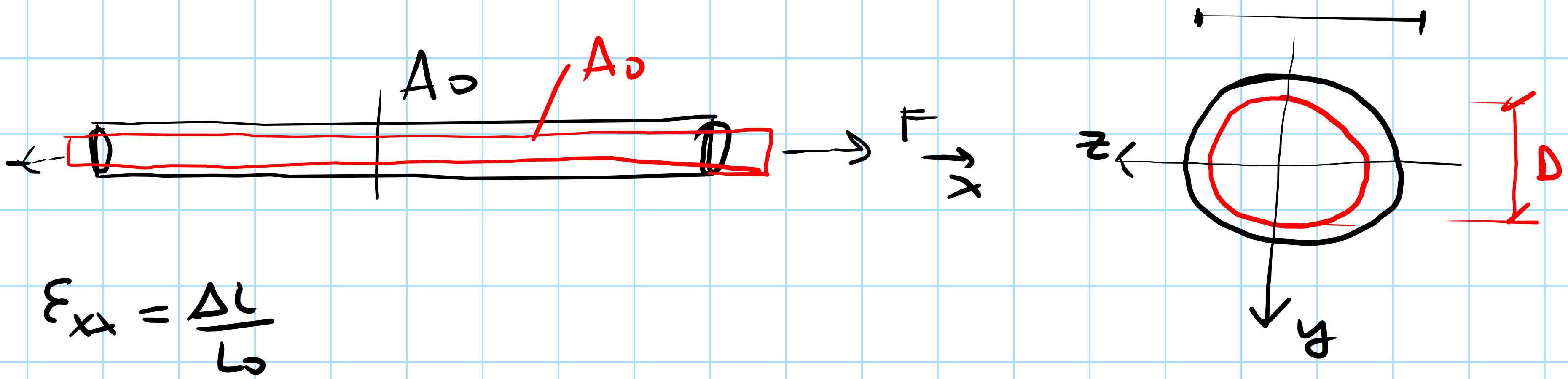
OUTPUT: DIFFERENT STRAINS



$\sigma = E\varepsilon$, E:

- 210 GPa STEEL
- 70 " ALUMINIUM
- 10 MPa POLYURETHAN

let us introduce the Poisson ratio ν



$$\varepsilon_{xx} = \frac{\Delta L}{L_0}$$

$$\varepsilon_{zz} = \varepsilon_{yy} = \frac{\Delta D}{D_0} = \frac{D - D_0}{D_0} < 0$$

$$\boxed{\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}}$$

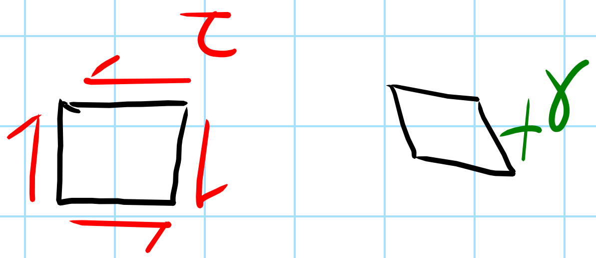
POISSON RATIO [-]
(DEGREE OF SHRINKING)

$\nu = 0.25 \div 0.35$ STEEL

$\nu \to 0.5$ RUBBER

$\nu < 0$ AUXETIC MATERIALS

ANOTHER MODULUS THAT CAN BE FOUND IS: SHEAR MODULUS (G) μ



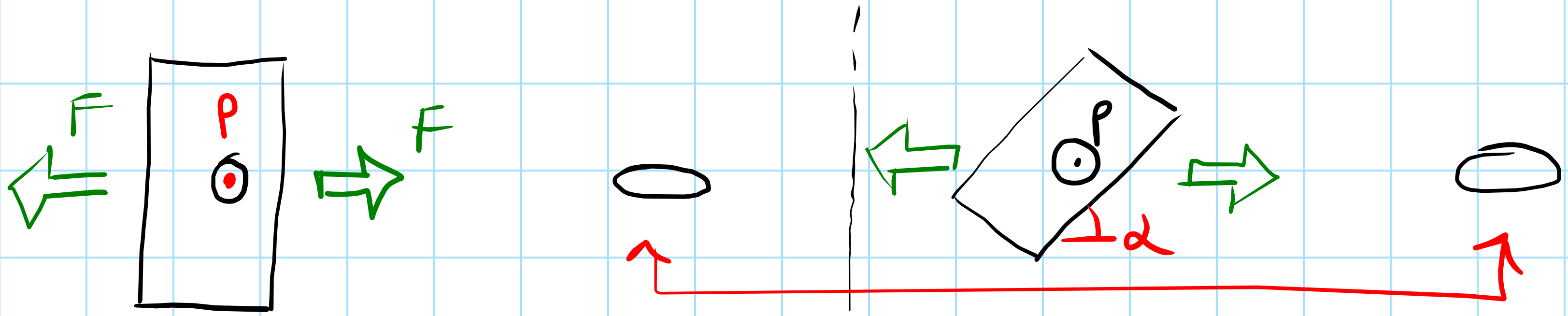
$$\tau = G\gamma$$

FOR ISOTROPIC MATERIALS

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

(ONLY TWO MODULI ARE INDEPENDENT)

CONCEPT OF ISOTROPY



ISOTROPY : SAME DEFORMATION $(\forall \alpha)$

CONCEPT OF HOMOGENEITY : A BODY IS HOMOGENEOUS IF $\underset{\sim}{\sigma}(P) = f(\underset{\sim}{\varepsilon}(P))$

IS VALID $\forall P \in \Omega$

LINEAR ELASTICITY : THE MOST GENERAL FORM OF CONST. LAW IN LINEAR ELASTICITY

$\underset{\sim}{\sigma} = \mathbb{C}\,\underset{\sim}{\varepsilon} \rightarrow \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$. HOW MANY INDEPENDENT CONSTANTS ARE AT MOST IN $\mathbb{C}$? AS $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ AND $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^T \Rightarrow$

4TH ORDER TENSOR

$\mathbb{C}$ HAS 36 INDEPENDENT CONSTANTS

ISOTROPIC LINEAR ELASTICITY : ONLY 2 CONSTANTS (E, ν) , (G, ν)

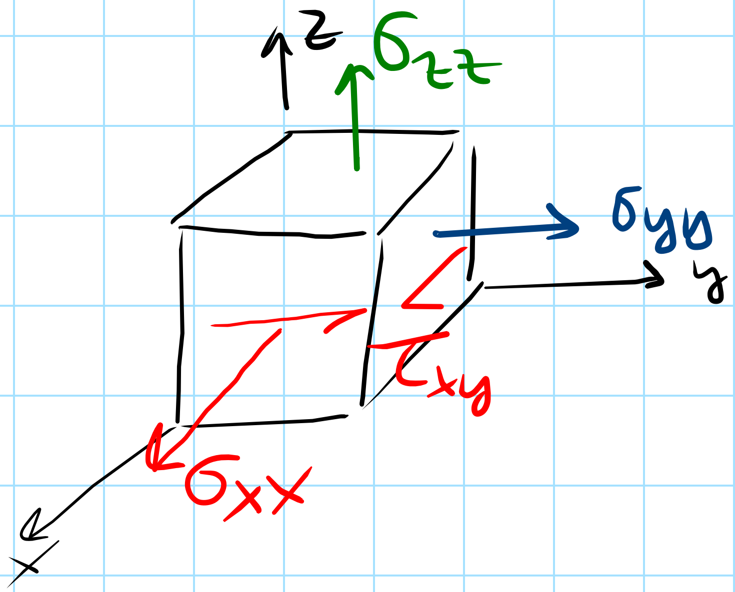
SOMETIMES IT IS USEFUL TO WRITE $\underset{\sim}{\varepsilon} = \mathbb{C}^{-1} \underset{\sim}{\sigma}$; OF COURSE $\mathbb{C}^{-1}$ EXISTS IF $\mathbb{C}$ IS NOT SINGULAR, BUT THIS IS NOT THE CASE FOR OUR MATERIALS.

WE CAN SHOW THAT $\mathbb{C}$ IS POSITIVE DEFINITE:

$$\mathbb{C}\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\varepsilon} > 0 \quad \forall \underset{\sim}{\varepsilon} \neq \underset{\sim}{0}$$

$$\mathbb{C}\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\varepsilon} = 0 \iff \underset{\sim}{\varepsilon} = \underset{\sim}{0}$$

ISOTROPIC LINEAR ELASTIC LAW



$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = -\nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \rightsquigarrow \varepsilon_{xy} = \frac{\tau_{xy}}{2G}, \quad \varepsilon_{xz} = \frac{\tau_{xz}}{2G}, \quad \varepsilon_{yz} = \frac{\tau_{yz}}{2G} \qquad \left(G = \frac{E}{2(1+\nu)}\right)$$

$$\underset{\sim}{\varepsilon} = \frac{1}{E}\left[(1+\nu)\,\underset{\sim}{\sigma} - \nu\left(\mathrm{tr}\,\underset{\sim}{\sigma}\right)\underset{\sim}{I}\right]$$

GENERALIZED HOOKE'S LAW

$$\hookrightarrow \varepsilon_{xx} = \frac{1}{E}\left[(1+\nu)\,\sigma_{xx} - \nu\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) 1\right]$$

$$\boxed{\underset{\sim}{\sigma} = 2G\,\underset{\sim}{\varepsilon} + \lambda\left(\mathrm{tr}\,\underset{\sim}{\varepsilon}\right)\underset{\sim}{I}} \qquad \lambda = f(G, E)$$

LAMÉ PARAMETERS