... (WE CONTINUE WITH GENERALIZED HOOKE'S LAW) CSM, 1/4/26

VOIGT REPRESENTATION

$\underline{\sigma}, \underline{\varepsilon}$: ALGEBRAIC VECTORS (6 COMPONENTS)

$(\underset{\sim}{\sigma} = \mathbb{C} \underset{\sim}{\varepsilon})$

$$\underbrace{\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix}}_{\underline{\varepsilon}} = \underbrace{\begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/2G & & \\ & 0 & & & 1/2G & \\ & & & & & 1/2G \end{bmatrix}}_{\mathcal{C}^{-1}} \underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}}_{\underline{\sigma}}$$

LET US STUDY THE RESTRICTIONS OF MATERIAL PARAMETERS (E, ν) ASSUMING THAT $\mathcal{C}^{-1}$ IS POSITIVE DEFINITE

① $\frac{1}{E} > 0 \longrightarrow \boxed{E > 0}$

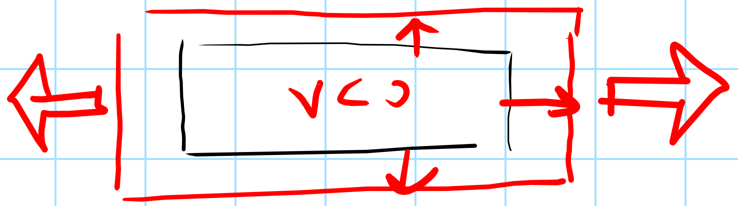
② $\frac{1}{E^2} - \frac{\nu^2}{E^2} > 0 \longrightarrow 1 - \nu^2 > 0, \quad \nu^2 < 1 \longrightarrow -1 < \nu < 1$

③ $\frac{1}{E^3} (1+\nu)^2 \underline{(1-2\nu)} > 0 \longrightarrow 1 - 2\nu > 0 \longrightarrow \nu < \frac{1}{2}$

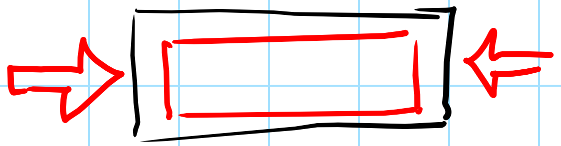
$\boxed{-1 < \nu < \frac{1}{2}}$

MATERIALS WITH $\nu < 0$ ARE ADMITTED: AUXETIC MATERIALS

WHAT DOES $\nu < 0$ MEAN?



LATERAL EXPANTION UPON PULLING



" SHRINKING UPON COMPRESSION

$\nu \to \frac{1}{2}$: INCOMPRESSIBLE MATERIALS (ELASTOMERS, RUBBERS)

$tr\,\underset{\sim}{\varepsilon}$: CHANGE IN VOLUME

PLAYING WITH THE HOOKE'S LAW

$$tr\,\underset{\sim}{\varepsilon} = \frac{1}{E}(1-2\nu)\,tr\,\underset{\sim}{\sigma}$$

$$tr\,\underset{\sim}{\varepsilon} = \frac{1}{E}(1-2\nu)\,3\sigma_m \quad \Big| \quad \sigma = E\,\varepsilon$$

$$\Rightarrow \sigma_m = \underset{\uparrow}{K}\,tr\,\underset{\sim}{\varepsilon}$$

BULK MODULUS

VOLUMETRIC RELATIONSHIP IN ISOTROPIC LINEAR ELASTICITY

IDENTIFYING THE TWO EQS, THEN

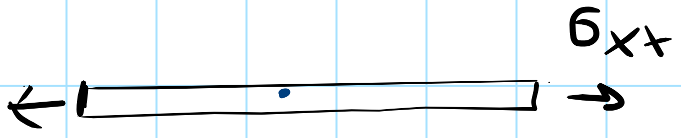
$$K = \frac{E}{3(1-2\nu)}$$

WHEN $\nu \to \frac{1}{2}$, $K \longrightarrow \infty$

THEN $tr\,\underset{\sim}{\varepsilon} = 0$: INCOMPRESSIBLE MATERIAL

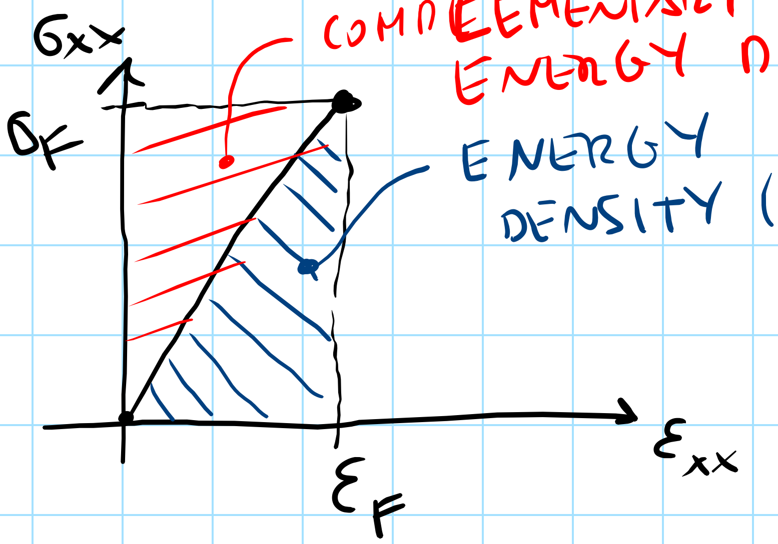
SOME CONSIDERATIONS ON STORED ENERGY IN AN ELASTIC BODY: THE ELASTIC POTENTIAL

1D:



$\sigma_{xx} \rightarrow \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, $\sigma_{xx} = E\,\varepsilon_{xx}$

$\left(\varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx}\right)$



COMPLEMENTARY ENERGY DENSITY (φ^c)

ENERGY DENSITY (PER UNIT VOLUME) : ELASTIC POTENTIAL (φ)

$$\varphi = \frac{1}{2}\sigma_F\varepsilon_F = \frac{1}{2}E\,\varepsilon_F^2 = \frac{1}{2}\frac{\sigma_F^2}{E}$$

$$[\varphi] = \left[\frac{F}{L^2}\right] = \left[\frac{\overset{\text{WORK}}{\boxed{FL}}}{\underset{\text{VOLUME}}{\boxed{L^3}}}\right]$$

$\left(\frac{J}{m^3}\right)$

Φ: TOTAL STORED ENERGY IN Ω : $\Phi = \int_V \varphi\, dV$

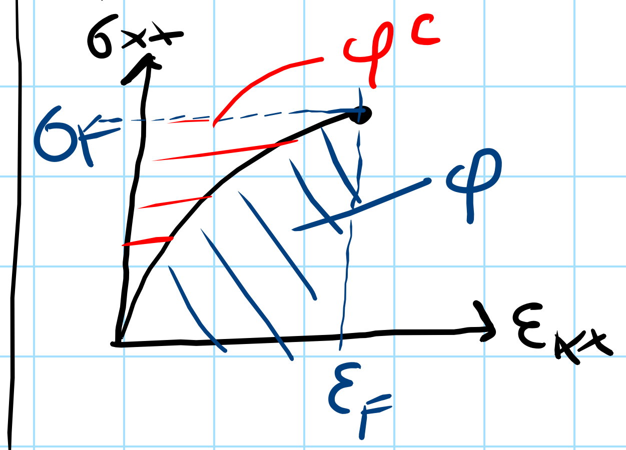
NOTE THAT IN LINEAR ELASTICITY

$\varphi = \varphi^c$ BUT IF I GO DEEP IN THE THEORY I DISCOVER THAT :

$$\varphi \rightarrow \varphi(\varepsilon_F) = \frac{1}{2}E\,\varepsilon_F^2$$

$$\varphi^c \rightarrow \varphi^c(\sigma_F) = \frac{1}{2}\frac{\sigma_F^2}{E}$$

HOWEVER, φ, φ^c ARE STILL VALID FOR NON-LINEAR ELASTICITY : $\varphi^c \neq \varphi$

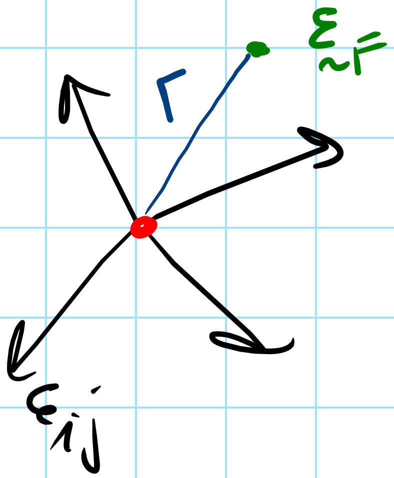


$$\varphi + \varphi^c = \sigma_F\varepsilon_F$$

IN ALL CASES

IN GENERAL (3D), IMAGINE TO REACH, AT A CERTAIN STAGE, THE STATUS $(\underset{\sim}{\sigma}_F, \underset{\sim}{\varepsilon}_F)$.

WHAT IS $\varphi(\underset{\sim}{\varepsilon}_F)$

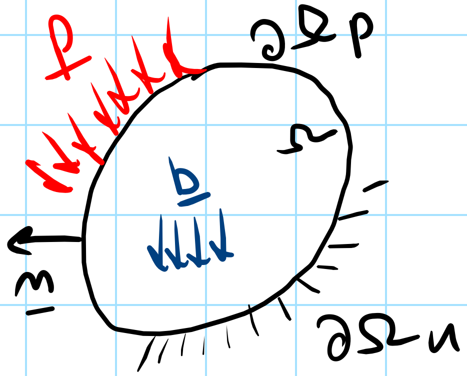


$$\varphi(\underset{\sim}{\varepsilon}_F) = \int_\Gamma \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}) \cdot d\underset{\sim}{\varepsilon} \longrightarrow \frac{1}{2} \mathbb{C} \underset{\sim}{\varepsilon}_F \cdot \underset{\sim}{\varepsilon}_F$$

Γ: PATH JOINING $\underset{\sim}{0}$ TO $\underset{\sim}{\varepsilon}_F$

(6-DIM SPACE)

LINEAR ELASTIC PROBLEM (L.E.P.)



$$\text{in } \boxed{\Omega} \begin{cases} \operatorname{div} \underset{\sim}{\sigma} + \boxed{\underline{b}} = \underline{0} & (3) \\ \underset{\sim}{\varepsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) & (6) \\ \underset{\sim}{\sigma} = \mathbb{C}\, \underset{\sim}{\varepsilon} & \underline{(6)} \end{cases}$$

15 EQS IN TOTAL

BOUNDARY CONDITIONS

$\underset{\sim}{\sigma}\,\underline{m} = \boxed{\underline{p}}$ on $\partial\Omega_p$

$\underline{u} = \boxed{\underline{\bar{u}}}$ on $\partial\Omega_u$ ($\underline{\bar{u}}$: ASSIGNED DISPL.)

HOW MANY UNKNOWNS?

$\underline{u}(p)$ (3)

$\underset{\sim}{\varepsilon}(p)$: (6)

$\underset{\sim}{\sigma}(p)$: $\underline{(6)}$

15 UNKN!

MATH ANALYS. HAS SHOWN THAT, UNDER REASONABLY HYPOTHESES OF REGULARITY, THE L.E.P. HAS A SOLUTION (EXISTENCE); MOREOVER, WITH A SIMPLE ARGUMENT, IT CAN BE SHOWN THAT, PROVIDED $\mathbb{C}$ POSITIVE DEFINITE, THE SOLUTION IS UNIQUE (KIRCHHOFF'S THEOREM).

LINEARITY ENSURES SUPERPOSITION EFFECT.

METHODS OF SOLUTION

– DISPLACEMENT METHOD (FOCUS ON $\underline{u}$) (NAVIER) (DIRECT METHOD)

STRONG FORM

$$\mathrm{div}\left[\frac{1}{2}\,\mathbb{C}\,(\nabla\underline{u}+\nabla\underline{u}^T)\right]+\underline{b}=\underline{0} \qquad (\Omega)$$

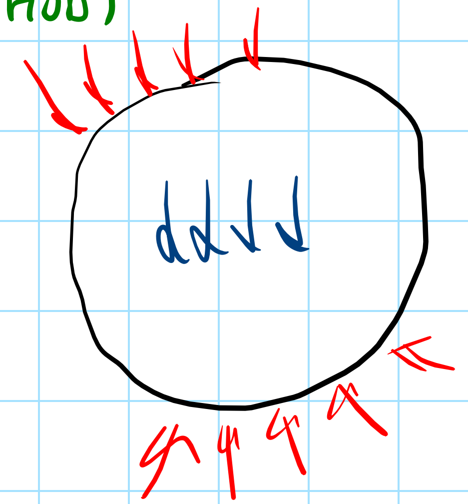
3 SCALAR EQS IN 3 UNKNOWNS

$$\left.\begin{aligned}\frac{1}{2}\,\mathbb{C}(\nabla\underline{u}+\nabla\underline{u}^T)\,\underline{n}&=\underline{p}\\ \underline{u}&=\underline{\bar{u}}\end{aligned}\right. \qquad (\partial\Omega)$$

STRONG FORM

(DIRECT METHOD)

– STRESS METHOD (FOCUS ON $\underset{\sim}{\sigma}$) (BELTRAMI-MICHELL)



$\underset{\sim}{\sigma}$ AS PRIMARY UNKNOWN ; VERY USEFUL WHEN $\partial\Omega_u=\phi$

(EQUILIBRATED BODY)

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IN GENERAL, THE L.E.P. IS A 'MIXED' PROBLEM WHERE EITHER DISPL OR STRESSES ARE UNKNOWN.

- OTHER METHODS BASED ON 'WEAK' FORMS (INTEGRAL FORMULATIONS) $\cong$ FUNCTIONALS
  - STATIONARITY OF TOTAL POTENTIAL ENERGY (DISPLACEMENTS ARE UNKNOWNS)

    COMPLEMENTARY POTENTIAL ENERGY (STRESS)

    HELLINGER-REISSNER FUNCTIONAL ($\underline{u}$, $\underset{\sim}{\sigma}$ UNKNOWNS : 9 COMPONENTS)

    HU-WASHIZU FUNCTIONAL ($\underline{u}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$: 15 COMPONENTS)

STAT. OF T.P.E. ($\Pi[\underline{u}]$)

$$\Pi[\underline{u}] = \underbrace{\frac{1}{2}\int_{\Omega} \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon}\, dV}_{\Phi\text{ : STORED ENERGY}} \overbrace{- \int_{\partial\Omega_p} \underline{p} \cdot \underline{u}\, dS - \int_{\Omega} \underline{b} \cdot \underline{u}\, dV}^{\text{POTENTIAL ENERGY OF EXT LOADS}}$$

$$\Pi[\underline{u}] = \frac{1}{2}\int_{\Omega} \frac{1}{2}\mathbb{C}(\nabla\underline{u} + \nabla\underline{u}^T) \cdot (\nabla\underline{u} + \nabla\underline{u}^T)\frac{1}{2}\, dV - \dots$$

PRINCIPLE OF STATIONARITY OF T.P.E.:

WITHIN THE SET OF DISPL. FIELD $\underline{u}$ KINEMATICALLY ADMISSIBLE ($\underline{u}(P)=\bar{\underline{u}}$ on $\partial\Omega_u$), THE ONE THAT SOLVE THE L.E.P. CORRESPONDS TO THE CONDITION

$$\boxed{\delta\Pi[\underline{u}]=0, \quad \forall \delta\underline{u}}$$

($\Longleftrightarrow$ SOLUTION OF L.E.P.)