PRINCIPLE OF STAT. OF T.P.E. $(\Pi[\underline{u}])$ CSM, 9/4/26

SKETCH OF THE PROOF

$$\Pi[\underline{u}] \to \Pi[\underline{u}+\delta\underline{u}] = \overbrace{\frac{1}{2}\int_\Omega \mathbb{C}\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\varepsilon}\, dV - \int_\Omega \underline{b}\cdot\underline{u}\, dV - \int_{\partial\Omega} \underline{p}\cdot\underline{u}\, dS}^{\Pi[\underline{u}]} + \overbrace{\int \mathbb{C}\underset{\sim}{\varepsilon}\cdot\delta\underset{\sim}{\varepsilon}\, dV - \int \underline{b}\cdot\delta\underline{u} - \int \underline{p}\cdot\delta\underline{u}}^{\delta\Pi}$$

$$+ \underbrace{\frac{1}{2}\int \mathbb{C}\delta\underset{\sim}{\varepsilon}\cdot\delta\underset{\sim}{\varepsilon}\, dV}_{\delta^2\Pi\ (>0)}$$

why $\delta\Pi=0$ is the solution of L.E.P.?

Because its expression is equivalent to a formulation of the theorem of virtual work.

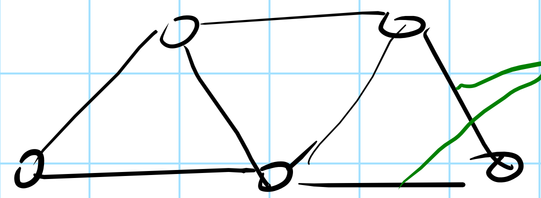
$L_{v\,int} - L_{v\,ext} = 0 \Rightarrow$ correspond to the solut. of the problem.

$$\Rightarrow \boxed{\delta\Pi = 0}$$

If we analyse $\delta^2\Pi$, we discover that $\delta^2\Pi > 0$ → $\delta\Pi=0$ corresponds to a minimum.

$\mathbb{C}$ positive definite

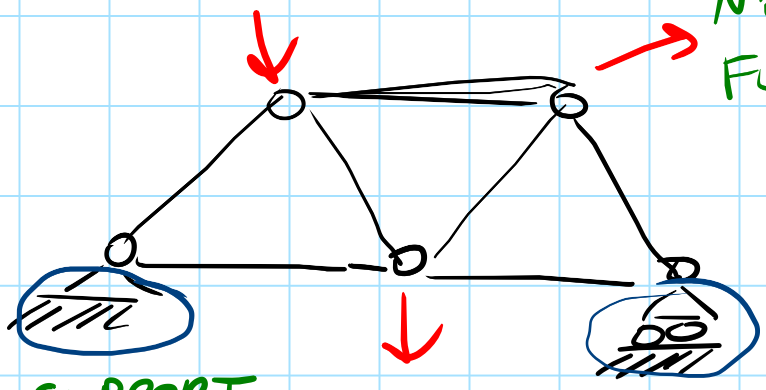
PLANE TRUSS STRUCTURE



BARS (MEMBERS OF THE STRUCTURE)

NODES (HINGES)

NODAL FORCES

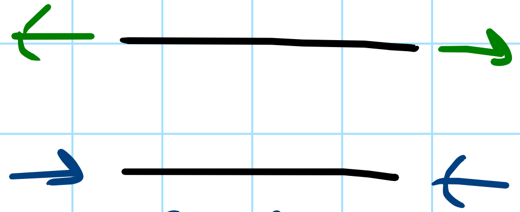


SUPPORT

ROLLER

EXTERNAL CONSTRAINTS

TENSION

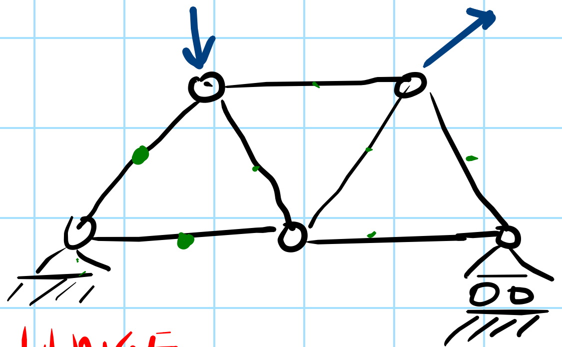


COMPRESS

(TIRANTE) TIE-ROD

STRUT (PUNTONE)

DEGREES OF FREEDOM OF A TRUSS STRUCTURE



HINGE (-2)

ROLLER (-1)

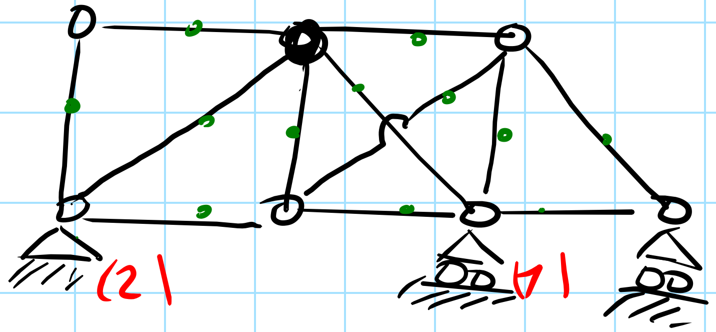
$5 \times 2 = 10$ FREE DOFS

N_{NODES} ↑ PLANE

BARS 7 (CONSTRAINTS)

3 EXTERNAL CONSTR.

10 TOT CONSTRAINTS



(2) (1) (1)

FREE DOFS : $7 \times 2 = 14$ 14

BARS 12

EXT CONSTR 4

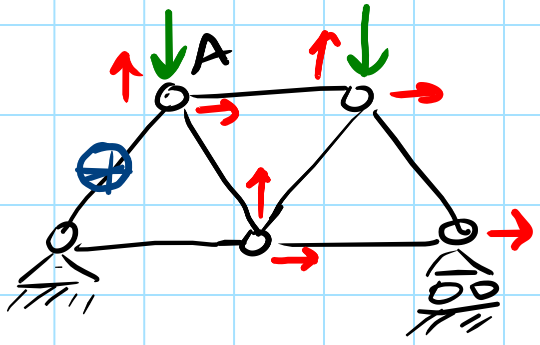
16 > 14

OVER CONSTRAINED STRUCTURE (REDUNDANT)

10 DOFS, 10 CONSTRAINTS → ISOSTATIC STRUCTURE

THE STRUCT. REMAINS FIXED.

TRUSS STRUCTURES AS A L.E.P.



IS IT POSSIBLE TO SOLVE THE L.E.P. BY CONSIDERING A FINITE NUMBER OF UNKNOWNS?

YES, BY CALCULATING FIRST THE NODAL DISPLACEMENTS. (7 IN OUR EXAMPLE)

⊕: BARS ARE ELASTIC.

AFTER HAVING CALCULATED THE 7 NODAL DISPLACEMENTS, WE FOCUS ON EACH BAR AND OBTAIN

- AXIAL STRAIN
- AXIAL FORCE THROUGH ELASTICITY.

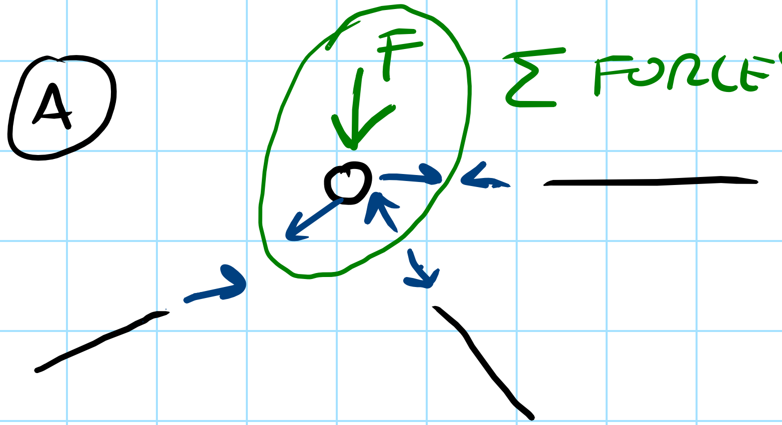
WHAT DOES 'SOLVING L.E.P.' MEAN?

TO GET STRESSES IN BARS AND IN EXTERNAL CONSTRAINTS; IN ADDITION, NODAL DISPLACEMENTS.

AFTER, WE CAN EXPLOIT NODAL EQUILIBRIUM TO OBTAIN EXTERNAL REACTION FORCES.

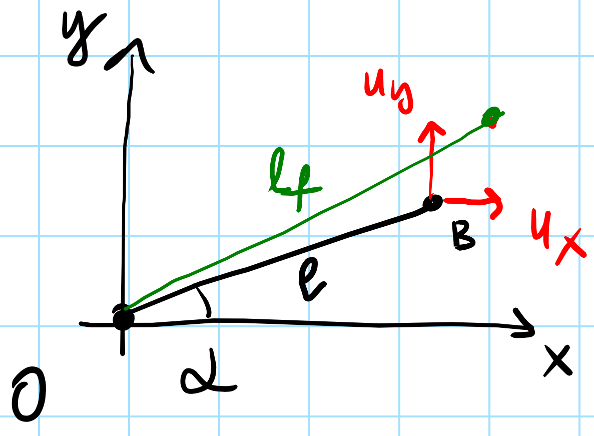
→ RECALL THAT EACH NODE MUST BE IN EQUILIBRIUM

A



$\sum$ FORCES $= 0$

INTRODUCTION TO DISPLACEMENT METHOD FOR TRUSS STRUCTURES

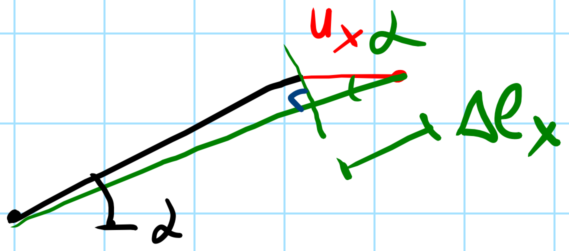


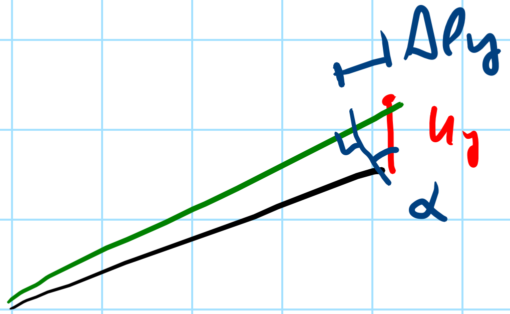
$$l=\sqrt{x_B^2+y_B^2} \; ; \; l_f=\sqrt{(x_B+u_x)^2+(y_B+u_y)^2}=l_f(u_x,u_y)$$

LET US EXPAND TO FIRST ORDER IN u_x, u_y:

$$l_f = l_f(0,0) + \frac{\partial l_f}{\partial u_x}\Big|_{(0,0)} u_x + \frac{\partial l_f}{\partial u_y}\Big|_{(0,0)} u_y + O(\quad)$$

$$l_f \simeq l + \left[\frac{1}{2\sqrt{\ }}\,2(x_B+u_x)\right]_{(0,0)} u_x + \left[\frac{1}{2\sqrt{\ }}\,2(y_B+u_y)\right]_{(0,0)} u_y = l + \underbrace{\overset{\cos\alpha}{\frac{x_B}{l}}\, u_x}_{\Delta l_x} + \underbrace{\overset{\sin\alpha}{\frac{y_B}{l}}\, u_y}_{\Delta l_y}$$





ACTUALLY, WE NEED $l_f - l = \Delta l = \Delta l_x + \Delta l_y \longrightarrow \boxed{N = \frac{EA}{l}\Delta l}$

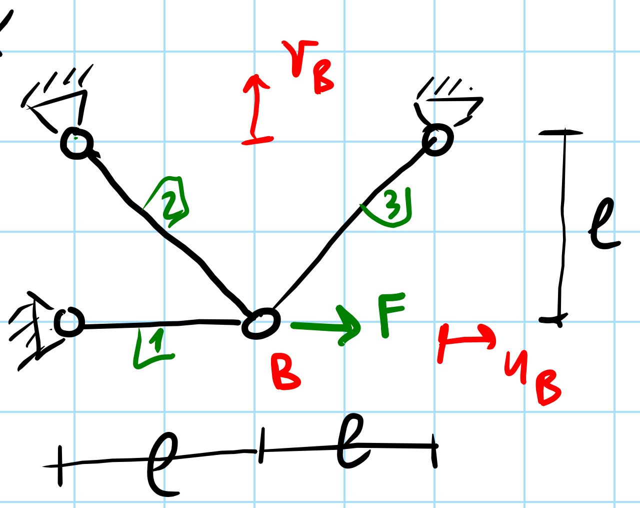
AXIAL FORCE

$\left(\sigma = E\varepsilon\right)$, $\sigma = \frac{N}{A}$, $\varepsilon = \frac{\Delta l}{l}$

K : STIFFNESS OF THE BAR.

THIS IS VALID ONCE u_x, u_y ARE KNOWN!

EX



u_B, v_B : PRIMARY UNKNOWNS | $N_i = \frac{EA_i}{l_i} \Delta l_i (u_B, v_B)$

$N_1 = \frac{EA}{l} \Delta l_1 \; ; \; N_2 = \frac{E\cancel{\sqrt{2}}A}{\cancel{\sqrt{2}}l} \Delta l_2 \; ; \; N_3 = \frac{E\cancel{\sqrt{2}}A}{\cancel{\sqrt{2}}l} \Delta l_3$

$K = \frac{EA}{l}$

E : CONST

$A_1 = A \; ; \; A_2 = A_3 = \sqrt{2} A$

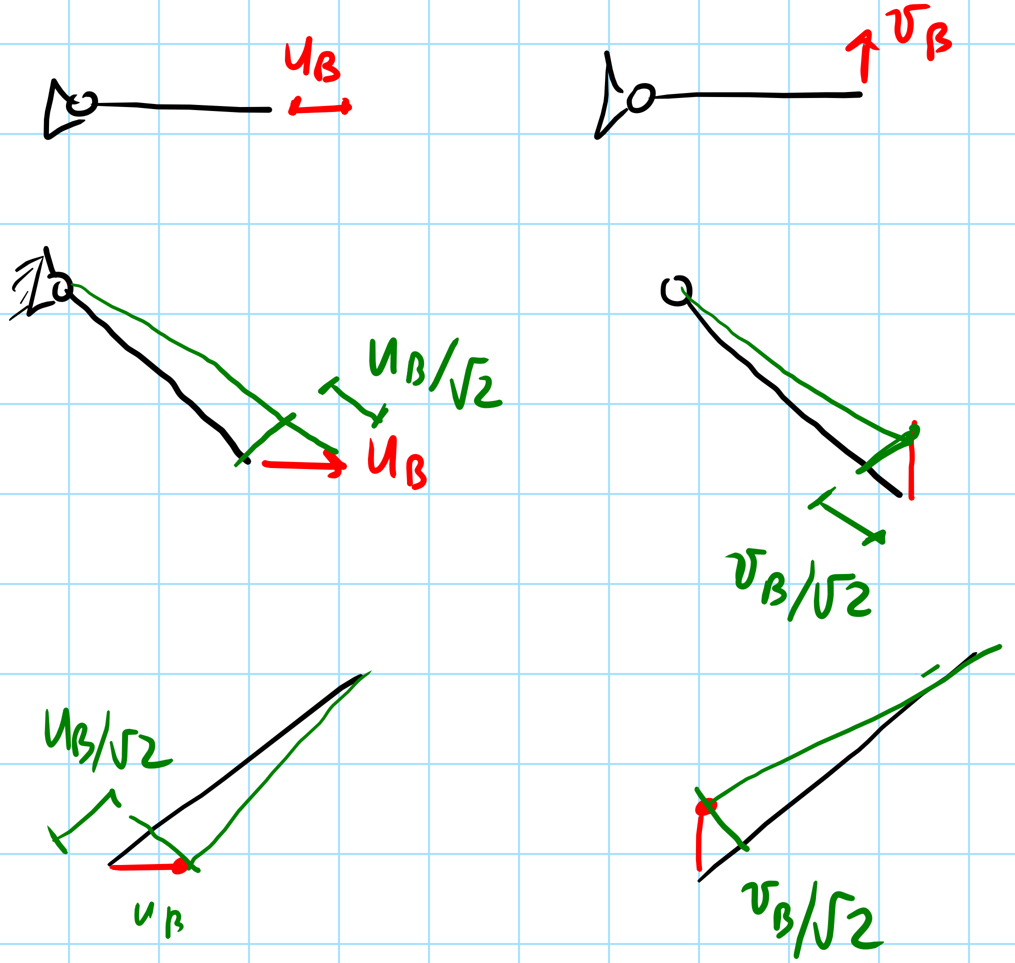
$l_1 = l \; ; \; l_2 = l_3 = \sqrt{2}\, l$

u_B v_B

$\Delta l_1)$

$\Delta l_2)$

$\Delta l_3)$



$\Delta l_1 = + u_B$

$\Delta l_2 = + \frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$

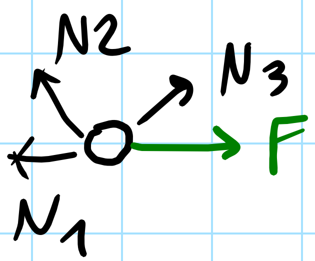
$\Delta l_3 = - \frac{u_B}{\sqrt{2}} - \frac{v_B}{\sqrt{2}}$

$N_1 = K \, (u_B)$

$N_2 = \frac{K}{\sqrt{2}} (u_B - v_B)$

$N_3 = \frac{K}{\sqrt{2}} (-u_B - v_B)$

NOW EQUILIBR OF NODE B MUST BE INVOKED;



$$\begin{cases} \sum \overset{+}{\rightarrow} \mathcal{F}_x = 0 \\ \sum \mathcal{F}_y = 0 \end{cases} \qquad \begin{cases} -N_1 - \dfrac{N_2}{\sqrt{2}} + \dfrac{N_3}{\sqrt{2}} + F = 0 \\ +N_2/\sqrt{2} + \dfrac{N_3}{\sqrt{2}} = 0 \end{cases} \qquad \begin{cases} -K u_B - \dfrac{K}{2}(u_B - v_B) + \dfrac{K}{2}(-u_B - v_B) + F = 0 \\ \dfrac{K}{2}(u_B - v_B) + \dfrac{K}{2}(-u_B - v_B) = 0 \end{cases}$$

NON-HOMOG. SYSTEM (2 UNKNOWNS)

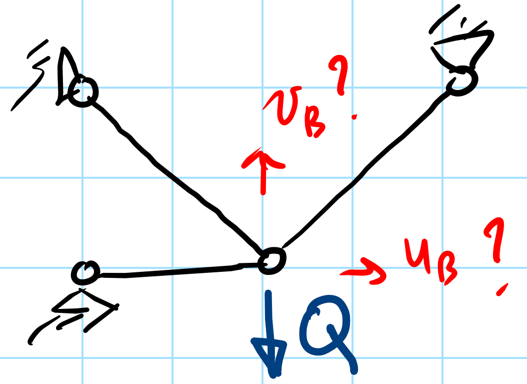
(1 SOLUTION!)

$$\underbrace{\begin{bmatrix} 2K & 0 \\ 0 & -K \end{bmatrix}}_{\substack{\text{STIFFNESS MATRIX OF} \\ \text{THE SYSTEM } \underset{\sim}{K}}} \underbrace{\begin{bmatrix} u_B \\ v_B \end{bmatrix}}_{\substack{\text{VECTOR OF} \\ \text{NODAL DISPL.}}} = \underbrace{\begin{bmatrix} F \\ 0 \end{bmatrix}}_{\substack{\text{VECTOR OF} \\ \text{KNOWN TERMS}}}$$

THE SOLUTION IS $v_B = 0$, $u_B = +F/2K$

NOTE THAT, IN GENERAL, THE STIFFNESS MATRIX IS SYMMETRIC AND DEFINITE POSITIVE. ITS INVERSE, IN LINEAR ELASTICITY, ALWAYS EXISTS!

QUESTION: (SOME STRUCTURE)



$$\begin{bmatrix} & K & \end{bmatrix} \begin{bmatrix} u_B \\ v_B \end{bmatrix} = \begin{bmatrix} 0 \\ -Q \end{bmatrix}$$

COMING BACK TO THE PREVIOUS PROBL:

$N_1 = K u_B = \frac{F}{2}$ THE ROD

$N_2 = \frac{K}{\sqrt{2}} (u_B - v_B) = \frac{K}{\sqrt{2}} \frac{F}{2K} = \frac{F}{2\sqrt{2}}$ THE ROD

$N_3 = \frac{K}{\sqrt{2}} (-u_B - v_B) = -\frac{K}{\sqrt{2}} \frac{F}{2K} = -\frac{F}{2\sqrt{2}}$ STRUT

LOOKING AT THE OBTAINED $\underset{\sim}{K}$,

LET US TRY TO UNDERSTAND THE MEANING OF ITS COMPONENTS

K_{11}? first line: $K_{11} u_B + K_{12} v_B = F$

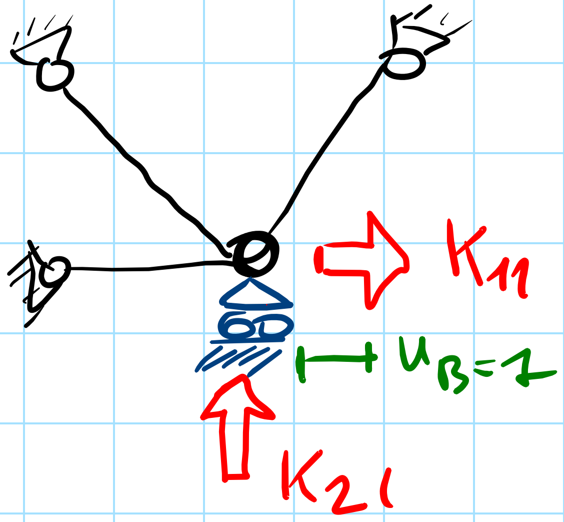
Let us take $u_B = 1$, $v_B = 0$: $K_{11} = F$

" " $v_B = 1$, $u_B = 0$: $K_{12} = F$

In general,

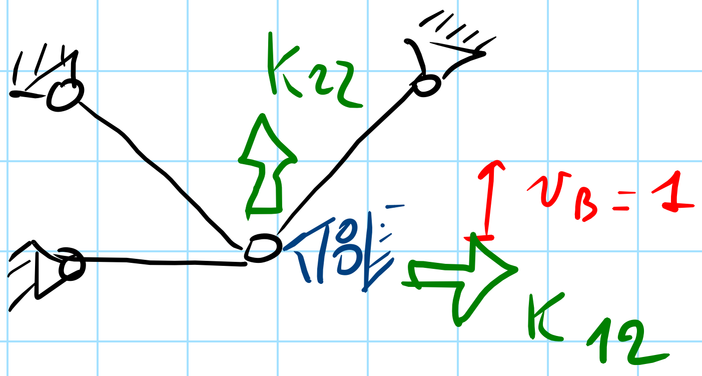
K_{ij} IS THE "FORCE" ASSOCIATED WITH DOF i WHEN DOF $j = 1$ AND THE REMAINING DOFS ARE ALL NULL

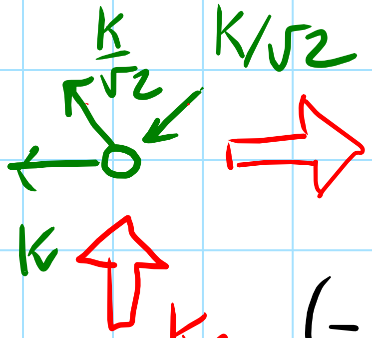
APPLYING THIS TO OUR EX:



$u_B = 1$ — K_{11}, K_{21}

$v_B = 0$

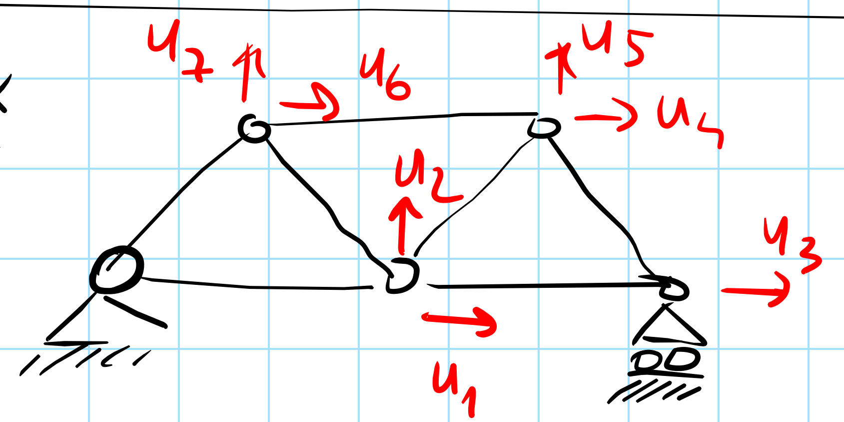




K_{11} $\left(K + \frac{K}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{K}{\sqrt{2}}\frac{1}{\sqrt{2}} = 2K\right)$

K_{21} $\left(-\frac{K}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{K}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0\right)$

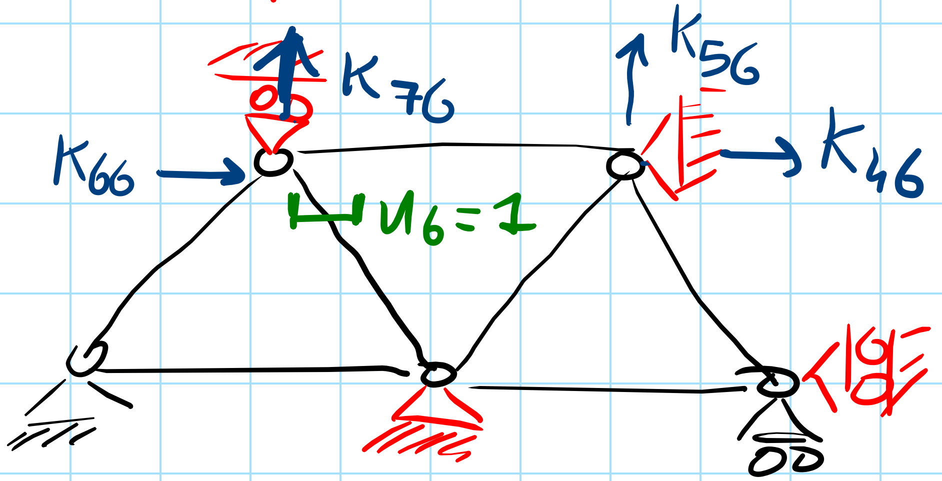
EX



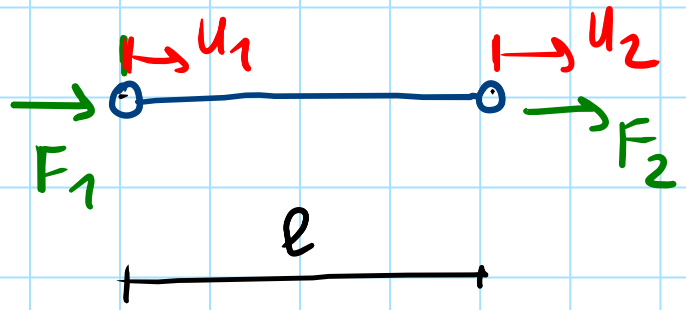
7 DOFS

$\hat{K}$: 7x7

K_{66}?



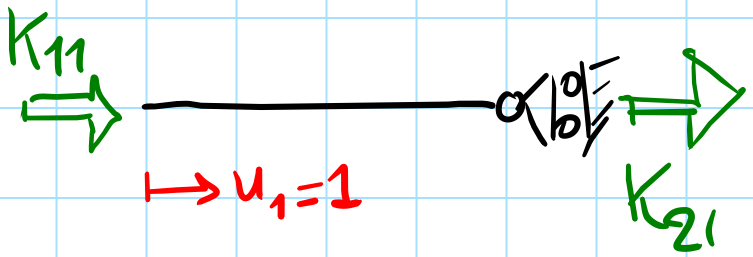
CAN WE DEFINE THE STIFFNESS MATRIX OF A SINGLE BAR?



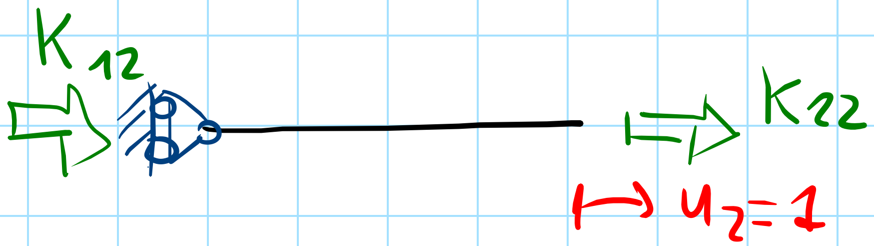
$$\frac{EA}{\ell}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

$$N=\frac{EA}{\ell}\Delta\ell$$

$\underset{\sim}{K} \longrightarrow$ STIFFN. MATRIX FOR A SINGLE MATRIX

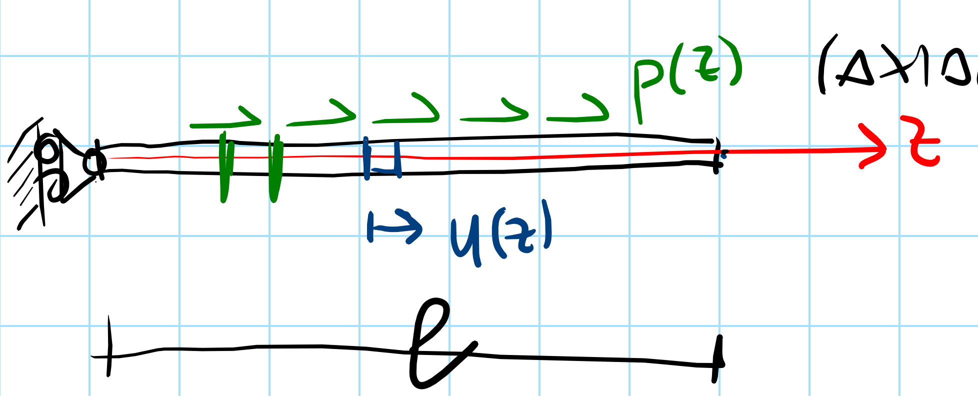


$$K_{11}=\frac{EA}{\ell}=-K_{21}$$



$$K_{22}=\frac{EA}{\ell}=-K_{12}$$

GOVERNING EQ FOR AN ELASTIC BAR:

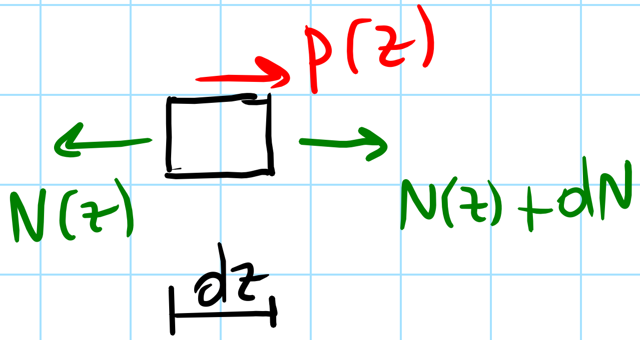


(AXIAL DISTRIBUTED FORCES)

UNKNOWNS: DISPLACEMENT MAP $u(z)$

AXIAL FORCE $N(z)$

• EQUILIBRIUM:



$\xrightarrow{+}$: $\cancel{N(z)} + dN - \cancel{N(z)} + p(z)\,dz = 0$

$$\frac{dN}{dz} = -p(z)$$

• COMPATIBILITY + ELASTICITY

$$\frac{du(z)}{dz} = \varepsilon = \frac{\sigma}{E} \rightarrow \frac{du(z)}{dz} = \frac{N}{EA} \quad \sigma$$

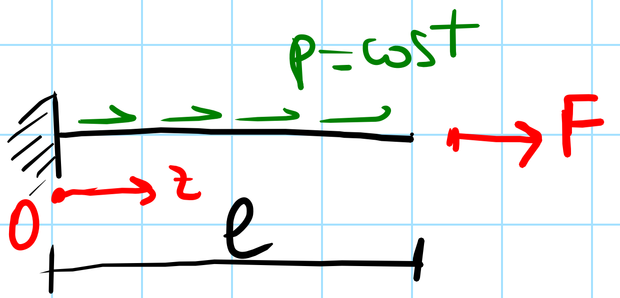
$$\boxed{\frac{du(z)}{dz} = \frac{N(z)}{EA}}$$

MERGING: (EA cost)

$$EA\,u'(z) = N(z) \rightarrow EA\,u''(z) = N'(z) \rightarrow \boxed{EA\,u''(z) = -p(z)}$$

"FINAL" GOVERNING EQ OF A BAR.

EX



$u(z)$?

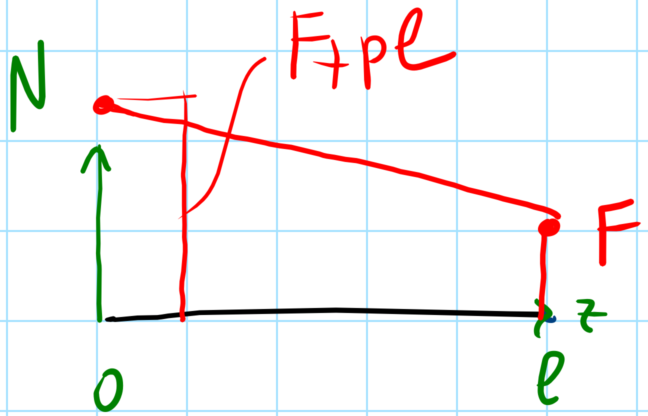
$N(z)$?

$$\begin{cases} EA\,u''(z) = -p \\ u(0)=0 \\ N(\ell)=+F \rightarrow EA\,u'(\ell)=F \end{cases}$$

$$u''(z) = -\frac{p}{EA}$$

$$u'(z) = -\frac{p}{EA}z + C_1 \rightarrow EA\,u'(\ell) = -p\ell + C_1 EA = F$$

$$C_1 = \frac{F+p\ell}{EA}$$



$$u(z) = -\frac{p}{EA}\frac{z^2}{2} + C_1 z + C_2 \rightarrow u(0)=0 \rightarrow C_2 = 0$$

$$\boxed{u(z) = -\frac{p}{EA}\frac{z^2}{2} + \frac{F+p\ell}{EA}z}$$

$$N(z) = EA\,u'(z) = -pz + F + p\ell = F + p(\ell - z)$$