

# Compatibility Conditions

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# Regular Surfaces

## Definition

A subset  $S \subset \mathbb{R}^3$  is a *regular surface* if, for every  $p \in S$  there exist an open set  $V \subset \mathbb{R}^3$  an open set  $U \subset \mathbb{R}^2$  and a map  $\mathbf{x}$  of  $U$  onto  $V \cap S \subset \mathbb{R}^3$  such:

- $\mathbf{x}$  is differentiable, i.e. if  $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$  with  $(u, v) \in U$ , then  $x, y, z$  have continuous partial derivatives of all orders in  $U$ ;
- $\mathbf{x} : U \rightarrow V \cap S$  is a homeomorphism;
- $d\mathbf{x}_q$  has rank 2 at every point  $q \in U$

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- $\mathbf{x} : U \rightarrow V \cap S$  is a homeomorphism;
- $d\mathbf{x}_q$  has rank 2 at every point  $q \in U$

$$T_p S = \{\mathbf{v} \in \mathbb{R}^3 \text{ tangent to } S \text{ at } p = \mathbf{x}(q)\} = d\mathbf{x}_q(\mathbb{R}^2)$$

# Induced scalar product and First Fundamental Form

$$\langle \cdot, \cdot \rangle_{\mathbb{R}^3} |_{T_p S \times T_p S} = \langle \cdot, \cdot \rangle_p$$

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$$I_p \leftrightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

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# Gauss Map

$$N(p) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} \Big|_q \quad p = \mathbf{x}(q)$$

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$\{\mathbf{x}_u, \mathbf{x}_v, N\}$  is a (moving) frame of  $\mathbb{R}^3$

# Curves on a Surface

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A *smooth curve* on a surface  $S$  is a smooth map

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The curve is *regular* if  $\alpha'(t) \neq 0$  for all  $t \in I$ .

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If  $\alpha$  is parametrized by arc length,  $\|\alpha'(s)\| = 1$  for all  $s$ .

# Curves on a Surface

If  $\alpha_1 : I \subset \mathbb{R} \rightarrow S$  and  $\alpha_2 : I \subset \mathbb{R} \rightarrow S$  are two regular curves on  $S$  which intersect at  $p = \alpha_1(t_0) = \alpha_2(t_0)$ , then if  $\vartheta$  is the angle between  $\alpha_1'(t_0)$  and  $\alpha_2'(t_0)$  is such that

$$\cos(\vartheta) = \frac{\langle \alpha_1'(t_0), \alpha_2'(t_0) \rangle_p}{\|\alpha_1'(t_0)\| \cdot \|\alpha_2'(t_0)\|}$$

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In particular the angle  $\varphi$  between line coordinates

$\alpha_1(t) = \mathbf{x}(u_0 + t, v_0)$   $\alpha_2(t) = \mathbf{x}(u_0, v_0 + t)$  at  $p = \mathbf{x}(u_0, v_0)$  is such that

$$\cos(\varphi) = \frac{\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p}{\|\mathbf{x}_u\| \cdot \|\mathbf{x}_v\|} = \frac{F}{\sqrt{EG}}$$

$\mathbf{x} : U \rightarrow V \cap S$   $R \subset \mathbf{x}(U)$   $Q \subset \mathbb{R}^2$  such that  $R = \mathbf{x}(Q)$

$\mathbf{x} : U$  onto  $V \cap S$   $R \subset \mathbf{x}(U)$   $Q \subset \mathbb{R}^2$  such that  $R = \mathbf{x}(Q)$

We define area of  $R$  to be

$$\int \int_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| dudv = \int \int_Q \sqrt{EG - F^2} dudv$$

# Differentiable functions

## Definition

$f : S \rightarrow S'$  is said to be *differentiable* at  $p \in \mathbf{x}(U)$  if there exists a local coordinate neighborhood  $\mathbf{y}(U')$  of  $f(p)$  in  $S'$  such that

$$\mathbf{y} \circ f \circ \mathbf{x}^{-1} : U \subset \mathbb{R}^2 \rightarrow U' \subset \mathbb{R}^2$$

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$$df_p : T_p S \rightarrow T_{f(p)} S'$$

is a linear operator

## Second Fundamental Form and Curvatures

In particular, since  $N : S \cap V \rightarrow S^2$

$$dN_p : T_p S \rightarrow T_{N(p)} S^2 \simeq T_p S$$

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$dN_p$  has two real eigenvalues,

$$dN_p(\mathbf{e}_1) = -k_1 \mathbf{e}_1 \quad dN_p(\mathbf{e}_2) = -k_2 \mathbf{e}_2,$$

called *principal curvatures* of  $S$  at  $p$ .

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## Second Fundamental Form and Curvatures

If  $\mathbf{v} = \mathbf{e}_1 \cos \vartheta + \mathbf{e}_2 \sin \vartheta$  with  $\mathbf{e}_1$  and  $\mathbf{e}_2$  principal directions of  $S$  at  $p$ , then

$$II_p(\mathbf{v}) = k_1 \cos^2 \vartheta + k_2 \sin^2 \vartheta \quad (\text{Euler's Formula})$$

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If  $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$  is a regular curve such that  $\alpha'(0) = p$  and  $\mathbf{v} = \alpha'(0) = \mathbf{x}_u u'(0) + \mathbf{x}_v v'(0) \in T_p S$ , then

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$$II_p(\alpha'(0)) = -\langle N_u, \mathbf{x}_u \rangle u'(0)^2 - 2\langle N_u, \mathbf{x}_v \rangle u'(0)v'(0) - \langle N_v, \mathbf{x}_v \rangle v'(0)^2$$

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$$f := -\langle N_u, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{uv} \rangle = -\langle N_v, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{vu} \rangle$$

$$g := -\langle N_v, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{vv} \rangle$$

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$$II_p \leftrightarrow \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

## Second Fundamental Form and Curvatures

On the other hand, since  $N_u \perp N$ ,  $N_v \perp N$

$$N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v$$

$$N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v$$

or

$$dN = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

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$$-e = \langle N_u, \mathbf{x}_u \rangle = a_{11}E + a_{21}F$$

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$$-f = \langle N_u, \mathbf{x}_v \rangle = a_{11}F + a_{21}G$$

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$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

or

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$$\frac{1}{2} \operatorname{tr} dN = \frac{1}{2} (a_{11} + a_{22}) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = \frac{k_1 + k_2}{2} := H$$

Mean Curvature

# Compatilby Equations

$$\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + L_1 N$$

$$\mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + L_2 N$$

$$\mathbf{x}_{vu} = \Gamma_{21}^1 \mathbf{x}_u + \Gamma_{21}^2 \mathbf{x}_v + \tilde{L}_2 N$$

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$$L_1 = e$$

$$L_2 = \tilde{L}_2 = f$$

$$L_3 = g$$

$\Gamma_{lm}^k$  Christoffel symbols

# Compatibilty Equations

$$\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + eN \rightarrow \begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = 1/2 E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - 1/2 E_v \end{cases}$$

$$\mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + fN \rightarrow \begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = 1/2 E_v \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = 1/2 G_v \end{cases}$$

$$\mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + gN \rightarrow \begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - 1/2 G_u \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = 1/2 G_v \end{cases}$$

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Since  $EG - F^2 \neq 0$ , one can express the Christoffel symbols in terms of  $E, F, G$  and their derivatives!

$$(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = 0$$

$$(\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v = 0$$

$$N_{uv} - N_{vu} = 0$$

$$A_1 \mathbf{x}_u + B_1 \mathbf{x}_v + C_1 N = 0$$

$$A_2 \mathbf{x}_u + B_2 \mathbf{x}_v + C_2 N = 0$$

$$A_3 \mathbf{x}_u + B_3 \mathbf{x}_v + C_3 N = 0$$

where  $A_l, B_l, C_l$  are functions of  $E, F, G, e, f, g$  and their derivatives.

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where  $A_l, B_l, C_l$  are functions of  $E, F, G, e, f, g$  and their derivatives. Since  $\{\mathbf{x}_u, \mathbf{x}_v, N\}$  are linearly independent, it turns out that  $A_l = 0, B_l = 0, C_l = 0 \quad l = 1, 2, 3$ .

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$$A_2 \mathbf{x}_u + B_2 \mathbf{x}_v + C_2 N = 0$$

$$A_3 \mathbf{x}_u + B_3 \mathbf{x}_v + C_3 N = 0$$

where  $A_l, B_l, C_l$  are functions of  $E, F, G, e, f, g$  and their derivatives. Since  $\{\mathbf{x}_u, \mathbf{x}_v, N\}$  are linearly independent, it turns out that  $A_l = 0, B_l = 0, C_l = 0 \quad l = 1, 2, 3$ . For instance

$$A_1 = 0 \iff (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 = F \cdot K$$

Gauss equation

# Theorema Egregium

## Definition

A diffeomorphism  $\psi : S \rightarrow S'$  is an isometry if

$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_p = \langle d\psi_p(\mathbf{w}_1), d\psi_p(\mathbf{w}_2) \rangle_{\psi(p)}$  for any  $\mathbf{w}_1, \mathbf{w}_2 \in T_p S$  and any  $p \in S$ .

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## Theorem

*The Gaussian curvature  $K$  of a surface depends only on the first fundamental form and therefore is invariant by (local) isometries.*

# Mainardi Codazzi equations

From

$$C_1 = 0 \iff e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2)_v - \Gamma_{11}^1 - g\Gamma_{11}^2$$

$$C_2 = 0 \iff f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2)_v - \Gamma_{12}^1 - g\Gamma_{12}^2$$

Mainardi Codazzi equations

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If  $F = 0$  and  $f = 0$ , the Mainardi Codazzi equations become

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$$e_v = e\Gamma_{12}^1 - g\Gamma_{11}^2$$

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and since

$$\Gamma_{11}^2 = -\frac{1}{2} \frac{E_v}{G} \quad \Gamma_{12}^1 = -\frac{1}{2} \frac{E_v}{E}$$

$$\Gamma_{22}^1 = -\frac{1}{2} \frac{G_v}{E} \quad \Gamma_{12}^2 = \frac{1}{2} \frac{G_v}{G}$$

we have

$$e_v = \frac{E_v}{2} \left( \frac{e}{E} + \frac{g}{G} \right) \quad g_u = \frac{G_u}{2} \left( \frac{e}{E} + \frac{g}{G} \right)$$

# Gauss equation

If  $F = 0$

$$K = -\frac{1}{2\sqrt{EG}} \left[ \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right]$$

# Theorem of Bonnet

## Theorem

Assume  $\mathcal{E}$ ,  $\mathcal{G}$ ,  $\mathcal{F}$ ,  $e$ ,  $g$ ,  $f$  are differentiable functions in an open set  $V \subset \mathbb{R}^2$  such that  $\mathcal{E} > 0$ ,  $\mathcal{G} > 0$ ,  $\mathcal{E}\mathcal{G} - \mathcal{F}^2 > 0$  and they (formally) satisfy Gauss and Mainardi Codazzi equations.

Then, for every  $q \in V$  there exists a neighborhood  $U \subset V$  of  $q$  and a diffeomorphism  $\mathbf{x} : U \rightarrow \mathbf{x}(U) \subset \mathbb{R}^3$  such that for the regular surface  $\mathbf{x}(U)$  the coefficients  $E, F, G, e, f, g$  of the first and second fundamental forms we have  $\mathcal{E} = E$ ,  $\mathcal{F} = F$ ,  $\mathcal{G} = G$ ,  $e = e$ ,  $f = f$ ,  $g = g$ . Furthermore, if  $U$  is connected and if  $\tilde{\mathbf{x}} : U \rightarrow \tilde{\mathbf{x}}(U) \subset \mathbb{R}^3$  is another diffeomorphism satisfying the same conditions, then there exist a translation  $T$  and a linear orthogonal transformation  $\rho$  of  $\mathbb{R}^3$  such that

$$\tilde{\mathbf{x}} = T \circ \rho \circ \mathbf{x}.$$