Lesson 2 - 26/02/2026

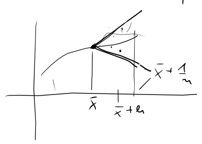
from the last lecture

$D = \{ f \in \mathcal{C}([0,1]),\ \exists \bar{x} \in]0,1[\text{ s.t. } f \text{ is differentiable at } \bar{x} \}$

$\widetilde{D} = \{ f \in \mathcal{C}([0,1]) : \exists \bar{x} \in [0,1[\text{ s.t. } D^+ f(\bar{x}) \in \mathbb{R},\ D_+ f(\bar{x}) \in \mathbb{R} \}$

$D \subseteq \widetilde{D}$

$$C_n = \left\{ f \in \mathcal{C}([0,1]) : \exists \bar{x} \in [0, 1-\tfrac{1}{n}] : \forall h \in]0, \tfrac{1}{n}]\ \left| \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right| \leq n \right\}$$



i) $C_n \subseteq \widetilde{D}$ ok

ii) $\widetilde{D} \subseteq \bigcup_{n=1}^{+\infty} C_n$ (exercise)

$f \in \widetilde{D} \Rightarrow \exists \bar{x} : D^+ f(\bar{x}) = \ell \in \mathbb{R} \quad D_+ f(\bar{x}) = \ell' \in \mathbb{R}$

$\ell' \leq \ell$

$$\inf_{t>0} \left(\sup_{\bar{x} < x < \bar{x}+t} \frac{f(x) - f(\bar{x})}{x - \bar{x}} \right) = \ell$$

$\Downarrow$

$$\exists t \quad \sup_{\bar{x} < x < \bar{x}+t} \frac{f(x) - f(\bar{x})}{x - \bar{x}} < \ell + 1$$

$$\forall x \in]\bar{x}, \bar{x}+t[\qquad \frac{f(x) - f(\bar{x})}{x - \bar{x}} < \ell + 1$$

from the fact that $D_+ f(\bar{x}) = \ell'$

we have $\exists t'$ s.t. $\forall x \in]\bar{x}, \bar{x}+t'[$

$$\frac{f(x) - f(\bar{x})}{x - \bar{x}} > \ell' - 1$$

on $]\bar{x}, \bar{x}+t''[$

$$\left| \frac{f(x) - f(\bar{x})}{x - \bar{x}} \right| \leq K \Rightarrow \exists n \text{ st. } f \in C_n$$

$D \subseteq \bigcup_n C_n$ closed in $\mathcal{C}([0,1])$ (with the sup norm top.)

Th. C_n $\mathring{C}_n = \emptyset$

proof. C_n closed

I take $(f_k)_k$ in C_n s.t. $f_k \to f$ uniformly

I prove that $f \in C_n$

$$C_n = \left\{ f \in \mathcal{C} : \exists \bar{x} \in [0, 1-\tfrac{1}{n}] \text{ s.t. } \forall h \in]0, \tfrac{1}{n}],\ \left| \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right| \leq n \right\}$$

$f_k \in C_n$, $\exists \bar{x}_k$ st. .... $\bar{x}_k \in [0, 1-\frac{1}{n}]$

passing to a subsequence $x_k \to \bar{x}$

$(f_k)_k\ C_n$, $\quad \bar{x}_k \to \bar{x}$

$f_k \to f$ uniformly.

I take ε, I take $h \in]0, \frac{1}{n}]$

$(f_k)_k \; C_n,$ $\quad \bar{x}_k \to \bar{x}$

$f_k \to f$ unifmly.

$\exists$ take ε, $\exists$ take $h \in \,]0, \frac{1}{n}]$

$\exists\, \bar{k}$ s.t. $\forall k \geq \bar{k}$:

a) $|f(\bar{x}_k + h) - f(\bar{x} + h)| \leq \frac{\varepsilon h}{4}$

b) $|f(\bar{x}_k) - f(\bar{x})| \leq \frac{\varepsilon h}{4}$

c) $\|f_k - f\|_{L^\infty([0,1])} < \frac{\varepsilon h}{4}$

$$|f(\bar{x}+h) - f(\bar{x})| \leq \underbrace{|f(\bar{x}+h) - f(\bar{x}_k+h)|}_{\leq \varepsilon h/4} + \underbrace{|f(\bar{x}_k+h) - f_k(\bar{x}_k+h)|}_{\frac{\varepsilon h}{4}}$$

$$+ \, |f_k(\bar{x}_k+h) - f_k(\bar{x}_k)|$$

$$+ \underbrace{|f_k(\bar{x}_k) - f(\bar{x}_k)|}_{\frac{\varepsilon h}{4}} + \underbrace{|f(\bar{x}_k) - f(\bar{x})|}_{\frac{\varepsilon h}{4}}$$

$$\leq \frac{\varepsilon h}{4} + \frac{\varepsilon h}{4} + \underbrace{|f_k(\bar{x}_k+h) - f_k(\bar{x}_k)|}_{f_k \in C_n \;\leq\; nh} + \frac{\varepsilon h}{4} + \frac{\varepsilon h}{4}$$

at the end $\quad |f(\bar{x}+h) - f(\bar{x})| \leq nh + \varepsilon h = (n+\varepsilon) h$

$$\left|\frac{f(\bar{x}+h) - f(\bar{x})}{h}\right| \leq n + \varepsilon \implies \underline{\text{OK}}$$

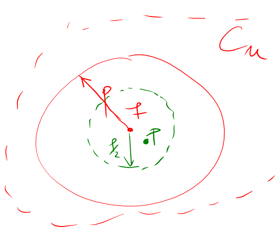
2) C_n have empty interior

by contradiction

suppose $\mathring{C}_n$ not empty

$\exists f \in C_n, \exists \rho > 0$ s.t. $B(f, \rho) \subseteq C_n$

$B(f,\rho) = \{g \in \mathcal{C} : \|f - g\|_{L^\infty} < \rho\}$



by Weierstrass

$\exists\, p$, polynomial s.t.

$\|p - f\|_{L^\infty} < \rho/2$

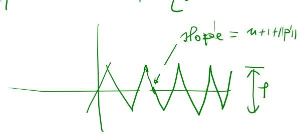
p polynomial $\Rightarrow \|p'\|_{L^\infty([0,1])} \in \mathbb{R}$

take $h : [0,1] \to \mathbb{R}$ s.t.

$\|h\|_{L^\infty} < \rho/2$

$\forall x \in [0,1]$, $|h'_+(x)| = n + 1 + \|p'\|_{L^\infty}$

h is a "sawtooth" function



consider $p + h$ $\qquad \|p - f\|_{L^\infty} \leq \rho/2$, $\|h\|_{L^\infty} < \rho/2$

$p + h \in B(f, \rho)$

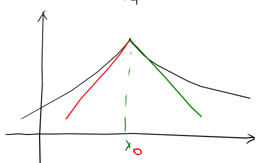
so $p + h \in C_n$ $\qquad$ but $|(p+h)'_+(x)|$

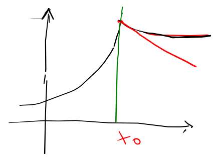
$\geq |h'_+(x)| - |p'| \geq n + 1$

$\Downarrow$

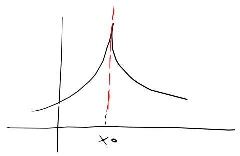
$h + p \notin C_n$

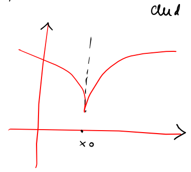
Def. x_0 is an angle if
$f'_+(x_0)$, $f'_-(x_0)$ exist
$f'_+(x_0) \neq f'_-(x_0)$ and one of them is in $\mathbb{R}$





Def x_0 is a cusp of $f'_+(x_0)$, $f'_-(x_0)$ infinite and different





Theorem let $f: [0,1] \to \mathbb{R}$, f continuous
let $A = \{ x \in]0,1[\text{ s.t. } x \text{ is angle or cusp} \}$
then A is at most countable

dim. consider $A^1 = \{ x \in]0,1[: f'_-(x) < f'_+(x) \}$
$A^2 = \{ x \in]0,1[: f'_-(x) > f'_+(x) \}$
$A = A^1 \cup A^2$

claim: A^1 is countable

let $x \in A^1$ then $\exists\, r \in \mathbb{Q}$ s.t. $f'_-(x) < r < f'_+(x)$
moreover $\exists\, t < x < s$ $t, s \in \mathbb{Q}$
s.t. $\forall y \in]t, x[$ $\quad \frac{f(y) - f(x)}{y - x} < r \Rightarrow f(y) - f(x) > r(y-x)$
$\forall y \in]x, s[$, $\quad \frac{f(y) - f(x)}{y - x} > r \Rightarrow f(y) - f(x) > r(y-x)$

In conclusion ~~I have~~ $(r, t, s) \in \mathbb{Q}$
s.t. $t < x < s$ and
for all $y \in]t, s[$, $y \neq x$,
$f(y) - f(x) > r(y-x)$ $\quad (*)$

I have $A^1 \longrightarrow \mathbb{Q}^3$
$x \mapsto (r, t, s)$

suppose A^1 is not (at most) countable
the above function is not injective

$\exists\, x_1 < x_2$ s.t. $x_1 \longrightarrow (r, t, s)$
$x_2 \longrightarrow (r, t, s)$

with $t < x_1 < x_2 < s$

I apply $(*)$ < write $x_1 = x$ and $x_2 = y$ (1)
write $x_2 = x$ and $x_1 = y$ (2)

(1) $f(x_2) - f(x_1) > r(x_2 - x_1) \iff f(x_1) - f(x_2) < r(x_1 - x_2)$

$f(x_1) - f(x_2) > r(x_1 - x_2)$ impossible