Lezione 3 / 27/02/2026

2. Lebesgue's differentiation theorem

Rem. lack of differentiability is caused by oscillation
on the contrary monotonicity guarantees diff. a.e.

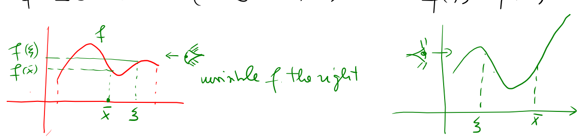
Th (Lebesgue) Consider $f : [a,b] \to \mathbb{R}$ monotone (increasing or decreasing)
then for almost every $x \in [a,b]$, f is differentiable at x

Remark continuity is not needed

Remark we will see the proof only in the case of f continuous (from Kolmogoroff - Fomin)

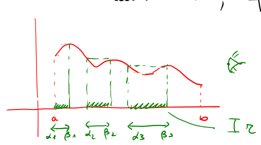
Remark invisible points.

def. Consider $f : [a,b] \to \mathbb{R}$, let $\bar{x} \in]a,b[$
$\bar{x}$ is said to be invisible from the right (left)
if $\exists \xi \in]\bar{x}, b]$ (or $\xi \in [a, \bar{x}[$) s.t. $f(\xi) > f(\bar{x})$



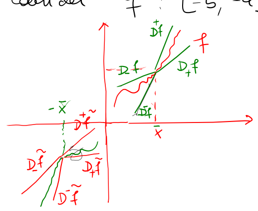
Lemma

let $f : [a,b] \to \mathbb{R}$ continuous
then $I_r = \{x \in]a,b[\,,\ x \text{ inv. from the right}\}$
I_r is an open set
so that $I_r = \bigcup_k]\alpha_k, \beta_k[$ pairwise disjoint intervals
and $\forall k$, $f(\alpha_k) \le f(\beta_k)$



($I_\ell = \{x \text{ inv. from left}\}$
$= \bigcup_k]\gamma_k, \delta_k[$ p. disjoint
$f(\gamma_k) \ge f(\delta_k)$)

Remark
suppose $f : [a,b] \to \mathbb{R}$
consider $\tilde{f} : [-b, -a] \to \mathbb{R}$, $\tilde{f}(x) = -f(-x)$



we have

$D_+ f(x) = D_- \tilde{f}(-x)$
$D^+ f(x) = D^- \tilde{f}(-x)$
$D_- f(x) = D_+ \tilde{f}(-x)$
$D^- f(x) = D^+ \tilde{f}(-x)$

symmetry

proof of Lebesgue's theorem

Remark f increasing then $D^+, D_+, D^-, D_- \ge 0$.

I claim that the proof will be a consequence of these two facts

1) for a.e. $x \in [a,b]$, $D^+ f(x) < +\infty$

2) for a.e. $x \in [a,b]$, $D^+ f(x) \le D_- f(x)$

let's see how to conclude from 1) and 2)

in particular 2) says that if f is increasing
$D^+ f(x) \le D_- f(x)$ for a.e. x
so that the same is valid for $\tilde{f}$

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so that

$$
\begin{array}{ccccccccc}
D_-\tilde{f}(-x) & \le & D^-\tilde{f}(-x) & \le & D_+\tilde{f}(-x) & \le & D^+\tilde{f}(-x) & \le & D_-\tilde{f}(x) \\
\| & & \| & & \| & & \| & & \| \\
D_+ f(x) & \le & D^+ f(x) & \le & D_- f(x) & \le & D^- f(x) & \le & D_+ f(x)
\end{array}
$$

$$\Downarrow$$

$$D_- f(x) = D^- f(x) = D_+ f(x) = D^+ f(x) \Rightarrow \exists f'(x)$$

and 1) for a.e. $x \in [a,b]$, $\quad D^+ f(x) < +\infty$

$$\Rightarrow \quad \underline{\exists f'(x) \in \mathbb{R} \text{ for almost every } x}$$

Let's prove 1) $\qquad f : [a,b] \to \mathbb{R}$, is increasing,

for a.e. $x \in [a,b]$, $\qquad D^+ f(x) < +\infty$

Let's prove 1) $f : [a,b] \to \mathbb{R}$, is increasing.

for a.e. $x \in [a,b]$, $D^+ f(x) < +\infty$

$$D^+ f(x) = \inf_{t>0} \left(\sup_{x<y<x+t} \frac{f(y)-f(x)}{y-x} \right) < +\infty$$

$$\exists \bar{t} \text{ s.t. } \sup_{x<y<x+\bar{t}} \frac{f(y)-f(x)}{y-x} < +\infty$$

suppose $A = \{x \in [a,b] : D^+ f(x) = +\infty\}$

I have to prove that $\lambda(A) = 0$

Lebesgue measure

let $x \in A$, let $C > 0$ then $D^+ f(x) > C$

$$\inf_{t>0} \left(\sup_{x<y<x+t} \frac{f(y)-f(x)}{y-x} \right) > C$$

$$\sup_{x<y<x+t} \frac{f(y)-f(x)}{y-x} > C$$

so that $\exists\, y > x$: $\frac{f(y)-f(x)}{y-x} > C$

but this means $\exists\, y > x$ s.t. $f(y) - f(x) > C(y-x)$

s.t. $f(y) - Cy > f(x) - Cx$

If $x \in A$, $\exists\, y > x$: $f(y) - Cy > f(x) - Cx$

x is invisible from the right for the function

$x \longmapsto f(x) - Cx$

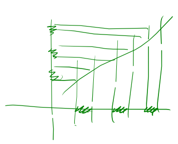
$$A \subseteq I_\varepsilon = \bigcup_k \,]\alpha_k, \beta_k[\quad \text{s.t. } f(\alpha_k) - C\alpha_k \leq f(\beta_k) - C\beta_k$$

$$\Downarrow$$

$$\beta_k - \alpha_k \leq \frac{1}{C}\left(f(\beta_k) - f(\alpha_k)\right)$$

$$\lambda(I_\varepsilon) = \sum_k (\beta_k - \alpha_k) \leq \sum_k \frac{1}{C}\left(f(\beta_k) - f(\alpha_k)\right)$$

$$\leq \frac{1}{C}\Big(\underbrace{\sum_k f(\beta_k) - f(\alpha_k)}_{\leq f(b) - f(a)}\Big)$$



$$A \subseteq I_{\varepsilon,C} \text{ and } \lambda(I_{\varepsilon,C}) \leq \frac{f(b)-f(a)}{C}$$

$$A \subseteq \bigcap_{C>0} (I_{\varepsilon,C}) \longrightarrow \lambda\Big(\bigcap_{C>0}(I_{\varepsilon,C})\Big) = 0$$

Summary f increasing / continuous $f : [a,b] \to \mathbb{R}$

$D^+ f(x) < +\infty$ for almost every x.

2) f increasing / continuous $f : [a,b] \to \mathbb{R}$

$D^+ f(x) \leq D_- f(x)$ for almost every x.

Lemma. Suppose λ^* is the outer Lebesgue's measure

let $A \subseteq [a,b]$

let $\rho \in [0,1[$

suppose that $\forall\,]\alpha,\beta[\subseteq\,]a,b[$

$\lambda^*(A \cap]\alpha,\beta[) \leq \rho(\beta - \alpha)$. Then A is measurable and $\lambda(A) = 0$

2) f increasing, continuous $f : [a,b] \to \mathbb{R}$

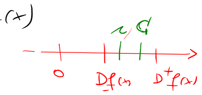
$D^+ f(x) \le D_- f(x)$ for almost every x.

let $B = \{x \in]a,b[: D^+ f(x) > D_- f(x)\}$

I want to prove that $\lambda(B) = 0$

let $x \in B$. then $D^+ f(x) > D_- f(x)$

$\exists\ r, q \in \mathbb{Q},\ 0 < r < q$ s.t.



$D^+ f(x) > q > r > D_- f(x)$

$B_{q,r} = \{x \in [a,b] : D^+ f(x) > q > r > D_- f(x)\}$

$$B = \bigcup_{\substack{q, r \in \mathbb{Q} \\ 0 < r < q}} B_{q,r}$$

sufficient to prove $\lambda(B_{q,r}) = 0$

let $x \in B_{q,r} \cap]\alpha, \beta[$ $\quad D_- f(x) < r < q < D^+ f(x)$

in particular $D_- f(x) < r$

$\sup_{\delta>0} \left(\inf_{x-\delta < y < x} \frac{f(y) - f(x)}{y - x} \right) < r$

in particular $\exists\ y < x$: $\frac{f(y) - f(x)}{y - x} < r$

$f(y) - f(x) > r(y - x)$

$f(y) - ry > f(x) - rx$

$x \in B_{q,r} \Rightarrow$ x is invisible from the left for $x \mapsto f(x) - rx$

$x \in I_r = \bigcup_k]\alpha_k, \beta_k[$

$f(\alpha_k) - r\alpha_k \ge f(\beta_k) - r\beta_k$

$f(\beta_k) - f(\alpha_k) \le r(\beta_k - \alpha_k)$

$x \in B_{q,r} \cap]\alpha_k, \beta_k[$

$D^+ f(x) > q \Rightarrow \exists\ y > x : \frac{f(y) - f(x)}{y - x} > q$

$f(y) - f(x) > q(y - x)$

$f(y) - qy > f(x) - qx$

x is inv. from the right for $x \mapsto f(x) - qx$

$x \in \bigcup_j]\alpha_{k,j}, \beta_{k,j}[$ s.t.

$f(\alpha_{k,j}) - q\alpha_{k,j} \le f(\beta_{k,j}) - q\beta_{k,j}$

$C(\beta_{k,j} - \alpha_{k,j}) \le f(\beta_{k,j}) - f(\alpha_{k,j})$

$\beta_{k,j} - \alpha_{k,j} \le \frac{1}{C}(f(\beta_{k,j}) - f(\alpha_{k,j}))$

$(B_{c,r} \cap]\alpha, \beta[) \subseteq \bigcup_k \left(\bigcup_j]\alpha_{k,j}, \beta_{k,j}[\right)$

$$\sum_k \left(\sum_j (\beta_{k,j} - \alpha_{k,j}) \right) \le \sum_k \left(\sum_j \frac{1}{C} (f(\beta_{k,j}) - f(\alpha_{k,j})) \right)$$

$$\le \sum_k \frac{1}{C} \sum_j \left(f(\beta_{k,j}) - f(\alpha_{k,j}) \right)$$

$$\sum_k \Big(\sum_j \big(\beta_{k,j} - \alpha_{k,j} \big) \Big) \leq \sum_k \Big(\sum_j \frac{1}{C} \big(f(\beta_{k,j}) - f(\alpha_{k,j}) \big) \Big)$$

$$\leq \sum_k \frac{1}{C} \sum_j \big(f(\beta_{k,j}) - f(\alpha_{k,j}) \big)$$

$$\leq \sum_k \frac{1}{C} \big(f(\beta_k) - f(\alpha_k) \big)$$

$$\leq \sum_k \frac{c}{C} (\beta_k - \alpha_k)$$

$$\leq \frac{c}{C} \underbrace{\sum_k (\beta_k - \alpha_k)}_{\leq (\beta - \alpha)}$$

$B_{C,c} \cap]\alpha, \beta[$ is inside a set $\Big(\bigcup_k \big(\bigcup_j]\alpha_{k,j}, \beta_{k,j}[\big) \Big)$ of length $\leq \frac{c}{C} (\beta - \alpha)$

$$\lambda^*\big(B_{C,c} \cap]\alpha, \beta[\big) \leq \frac{c}{C} (\beta - \alpha) \Rightarrow \underline{\underline{\lambda(B_{C,c}) = 0}}$$

Rem. How to prove the theorem for functions which are not continuous?

Kolmogoroff: use the fact that a monotone function has limit from right and left to define in modified way the set of invisible ....

Hewitt-Stromberg: use Vitali's coverings

Exercise (Fubini's theorem on differentiation of series)

Th. Let $(f_n)_n$ a sequence of increasing functions on $[a,b]$. Suppose that $\forall x \in [a,b]$, $\sum_n f_n(x)$ is converging with sum $s(x)$.

Then 1) s is a.e. differentiable

2) $\sum_n f_n'(x)$ is a.e. converging

3) $\sum_n f_n'(x) = s'(x)$

(Ex. construct a function $f : [0,1] \to \mathbb{R}$, strictly increasing s.t. $f'(x) = 0$ a.e.)

Proof. I remark that it is not restrictive to have $f(x) \geq 0$ (if not consider $g(x) = f(x) - f(a)$)

$f_n(x)$ is increasing $\forall n$

then $s_n(x) = f_1(x) + \dots + f_n(x)$, s_n is increasing

Since $s(x) = \lim_n s_n(x)$ also s is increasing

so that, by Lebesgue, s is a.e. differentiable.

consider $s(x) = s_n(x) + r_n(x)$, where $r_n(x) = \sum_{j=n+1}^{+\infty} f_j(x)$ (increasing)

so that $s'(x) = s_n'(x) + r_n'(x)$ a.e.

moreover $r_n'(x) \geq 0$ a.e.

so $s'(x) \geq s_n'(x) = \underbrace{\sum_{j=1}^{n} f_j'(x)}_{\geq 0}$

so that $J'(x) = J_n'(x) + r_n'(x)$ a.e.

moreover $r_n'(x) \geq 0$ a.e.

so $J'(x) \geq J_n'(x) = \overbrace{\sum_{j=1}^{n} f_j'(x)}^{\geq 0}$

the series $\sum_n f_n'(x)$ is bounded from above by $J'(x)$

so it is convergent $\sum_n f_n'(x)$

it remains to prove that $\boxed{\sum_n f_n'(x) = J'(x)}$

i.e. $J_n'(x) \xrightarrow{n} J'(x)$

it is sufficient $J_{n_k}'(x) \to J'(x)$ for a subsequence

I know that $J_n(b) \longrightarrow J(b)$

so $\exists\, n_k$ s.t. $\sum_k \left(J(b) - J_{n_k}(b)\right) < +\infty$

so for every $x \in [a,b]$ $\sum_k \left(J(x) - J_{n_k}(x)\right) < +\infty$

(because $0 \leq J(x) - J_{n_k}(x) \leq J(b) - J_{n_k}(b)$)

I call $g_k(x) = J(x) - J_{n_k}(x)$

and g_k is increasing

$\sum_k g_k(x)$ is converging

I apply the first part of the theorem to (g_k)

$\sum_k g_k'(x)$ is converging a.e.

$\Rightarrow$ for a.e. x $\lim_k g_k'(x) = 0$

$\lim_k J'(x) - J_{n_k}'(x) = 0$

QED