Data Science for Insurance

Risk Measures based on loss distribution

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Risk Measurement

Risk measures associate a financial position with a real number, a statistical quantity, describing the conditional or unconditional loss distribution of the portfolio over some predetermined horizon

- in the unconditional approach we denote the df of the loss $L = L_{t+1}$ simply by F_L
- Risk measures attempt to quantify the amount of assets that an insurer needs to retain to meet obligations
- it is natural to base a measure of risk on the *right tail* of the loss distribution (e.g. VaR, ES)

Purposes

Computation of risk measures is crucial for many insurance applications:

- to determine a measure of riskiness of insured claims
- to compute the amount of capital needed as a buffer against (unexpected) future losses to satisfy a regulator (*risk capital*)
- as a tool in financial risk management

Pros and cons

(Pros) The quantification of risk associated with a given loss distribution has some advantages:

- the concept of a loss distribution allows for aggregation at different levels
- if estimated properly, the loss distribution may provide an accurate picture of the risk in a portfolio
- loss distributions can be compared across portfolios

Pros and cons

(Cons) Two major issues arise when working with loss distributions

- estimates of the loss distribution are based on past data
- the assumption of normality is unrealistic in many situations, hence alternative statistical models are often needed
- risk measures based on the loss distribution should be complemented by information from hypothetical scenarios

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Preliminaries

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Value-at-Risk (**VaR**) can be viewed as an extension of maximum loss, which takes into account the probability information in F_L, by means of a given probability level [Jorion (2007)]

Computation of VaR involves *quantiles* of the loss distribution, hence we recall definitions of the *generalized inverse* and *quantile function*

Generalized inverses and quantiles

Let F be a df on $\mathbb{R}$.

(i.) The generalized inverse of F

$$F^{\leftarrow}(y) := \inf\{x \in \mathbb{R} \colon F(x) \geq y\}$$

is called the quantile function of F

(ii.) For $\alpha \in (0,1)$, the α-quantile of F is given by

$$q_\alpha(F) := F^{\leftarrow}(\alpha) := \inf\{x \in \mathbb{R} \colon F(x) \geq \alpha\}$$

Remarks

- if X is a rv with df F we set $x_\alpha := q_\alpha(F)$
- if F is continuous and strictly increasing $x_\alpha = q_\alpha(F) = F^{-1}(\alpha)$, where F^{-1} is the ordinary inverse of F

Definition of VaR

Definition (VaR). Given some level $\alpha \in (0,1)$, Value-at-Risk (VaR) of a portfolio with loss L at level α is defined as

$$\mathrm{VaR}_\alpha = \mathrm{VaR}_\alpha(L) = F_L^{\leftarrow}(\alpha) := \inf\{\ell \in \mathbb{R} \colon F_L(\ell) \geq \alpha\} \tag{1}$$

that is, VaR is the smallest number ℓ such that
$1 - F_L = P(L > \ell) \leq 1 - \alpha$.

- VaR is simply the α-quantile of the loss distribution (typically, we compute $\mathrm{VaR}_{0.95}$, $\mathrm{VaR}_{0.99}$)
- Market-risk: $\Delta t = 10$ days; Credit/operational: Δt=1 year
- VaR gives no information about the severity of losses occurring with a probability less than $1 - \alpha$
- VaR needs to be estimated from data

Mean-VaR

Let μ be the mean of the loss distribution. The statistic

$$\text{VaR}_\alpha^m := \text{VaR}_\alpha - \mu$$

denotes the **mean-VaR** and is used for capital-adequacy purposes instead of ordinary VaR.

- if $\Delta t = 1$ day, then VaR_α^m is referred to as *daily earnings at risk*
- Example: in loan pricing one uses VaR_α^m to determine the economic capital needed as a buffer against unexpected losses in a loan portfolio

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VaR for exponential distributions

Consider an insurance loss random variable L with an exponential distribution having mean $\theta > 0$:

$$f(\ell) = \frac{1}{\theta}\exp(-\ell/\theta); \quad F_L(\ell) = 1 - e^{-\ell/\theta}, \text{ for } \ell > 0.$$

Given $\alpha \in (0,1)$, $\mathrm{VaR}_\alpha(L)$ must be the value ℓ_α satisfying

$$\alpha = F_L(\ell_\alpha) = P(L \le \ell_\alpha) = 1 - \exp\{-\ell_\alpha/\theta\}$$

Hence

$$\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha) = -\theta\log(1-\alpha).$$

Remark: the VaR of any continuous random variables is simply the inverse of the corresponding cdf.

VaR for normal and lognormal distributions

Suppose $L \sim N(\mu, \sigma^2)$. Then, for a fixed $\alpha \in (0,1)$

$$\mathrm{VaR}_\alpha(L) = \mu + \Phi^{-1}(\alpha)\,\sigma,$$

where Φ denotes the standard normal df and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ. Clearly, $P(L \leq \mathrm{VaR}_\alpha(L)) = \alpha$.

VaR for normal and lognormal distributions

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where Φ denotes the standard normal df and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ. Clearly, $P(L \leq \mathrm{VaR}_\alpha(L)) = \alpha$.

Remark: the VaR of a linear transformation is equivalent to the linear transformation of the VaR:
If $Z \sim N(0,1) \ \rightarrow \quad \mathrm{VaR}_\alpha(Z) = \Phi^{-1}(\alpha)$
If $X = \mu + Z\sigma \ \rightarrow \quad \mathrm{VaR}_\alpha(X) = \mu + \mathrm{VaR}_\alpha(Z)\sigma$

This is in general true as long as the transformation is strictly increasing

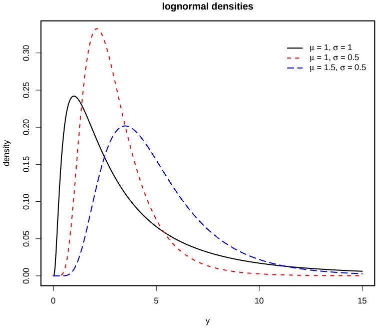
VaR for normal and lognormal distributions (cont)

Suppose $L \sim N(\mu, \sigma^2)$. Let $g(L) = Y = \exp(L)$. Then

$$Y \sim LogNormal(\mu, \sigma^2)$$

i.e. Y has a lognormal distribution with parameters $\mu \in \mathbb{R}$ and σ^2 ($\sigma > 0$).

For $\alpha \in (0,1)$, the VaR of $Y = e^L$ is



$$\mathrm{VaR}_\alpha(Y) = e^{\mathrm{VaR}_\alpha(L)} = \exp(\Phi^{-1}(\alpha)\,\sigma + \mu).$$

VaR for Student's t-distributions

Suppose $L^* = (L - \mu)/\sigma \sim t_\nu$, that is L^* has a "standard" Student's $t-$distribution with ν degrees of freedom (benchmark model in finance, usually $\nu = 3, 5$):

$$f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}(1 + x^2/\nu)^{-(\nu+1)/2}, \quad -\infty < x < \infty, \, \nu > 0,$$

where $\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt$.

VaR for Student's t-distributions (cont)

Let $L \sim t(\nu, \mu, \sigma^2)$ denotes the loss distribution.
We have $E(L) = \mu$ and $V(L) = \nu\sigma^2/(\nu - 2)$ when $\nu > 2$ (σ is not the standard deviation of the distribution.
We get

$$\mathrm{VaR}_\alpha = \mu + \sigma t_\nu^{-1}(\alpha),$$

where t_ν denotes the df of standard t with ν dof, and t_ν^{-1} is its inverse.

Illustration: VaR for Skew t-distributions

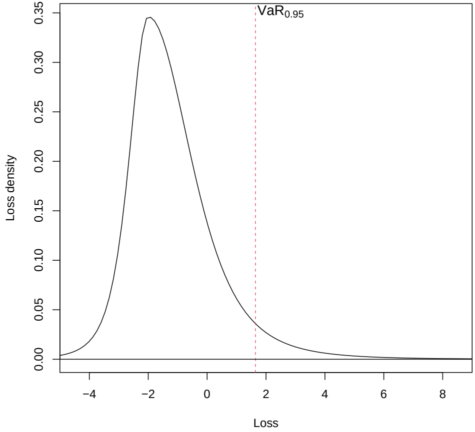


Figure: VaR ($\alpha = 0.95$) for a skew t_4-distribution

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Definition of Expected Shortfall (ES)

VaR does not reflect the extremal losses occurring beyond the $(1-\alpha) \times 100\%$ chance worst scenario. Expected shortfall was introduced by Artzner et al. (1997) (see also, Artzner et al. (1999))

Definition (ES). For a loss L with $E(|L|) < \infty$ and df F_L, the ES at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L) du \tag{2}$$

where $q_u(F_L) = F_L^{\leftarrow}(u) = \mathrm{VaR}_u(L)$ is the quantile function of F_L.

- ES is also known as *conditional value at risk* (CVaR)

Remarks on the ES measure

ES is obtained by averaging VaR, for all $u \geq \alpha$ (average loss when VaR is exceeded), hence ES depends on F_L and

$$\mathsf{ES}_\alpha(L) \geq \mathsf{VaR}_\alpha(L)$$

- If F_L is continuous, then VaR can be viewed as the *expected loss* that is incurred in the event that VaR is exceeded

$$\begin{aligned}\mathsf{ES}_\alpha(L) = \mathrm{E}(L|L > \mathsf{VaR}_\alpha(L)) &= \frac{\mathrm{E}(L, L > q_\alpha(L))}{1-\alpha} \\ &= \frac{1}{1-\alpha}\int_{q_\alpha(L)}^{\infty} \ell f(\ell) \mathrm{d}\ell\end{aligned}$$

- ES_α gives information about frequency and size of large losses (that occur when the VaR "bad times" threshold has been exceeded)

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ES for exponential distributions

Assume $L \sim Exp(1/\theta)$, so that $E(L) = \theta$.

For $\alpha \in (0, 1)$, we found $\mathrm{VaR}_\alpha(L) = -\theta \log(1 - \alpha)$. Hence, we obtain

$$\begin{aligned}
\mathrm{ES}_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)du \\
&= -\frac{\theta}{1-\alpha}\int_\alpha^1 \log(1-u)du \\
&= -\frac{\theta}{1-\alpha}\int_0^{1-\alpha} \log(y)dy \\
&= -\theta\log(1-\alpha) + \theta = \mathrm{VaR}_\alpha + \theta
\end{aligned}$$

ES for normal loss distributions

Suppose $L \sim N(\mu, \sigma^2)$. Then, for a fixed $\alpha \in (0,1)$

$$\mathrm{ES}_\alpha = \mu + \sigma E\left(\frac{L-\mu}{\sigma}\middle|\frac{L-\mu}{\sigma} \geq q_\alpha\left(\frac{L-\mu}{\sigma}\right)\right) = \mu + \sigma \mathrm{ES}_\alpha(L^*),$$

where $L^* := \dfrac{L-\mu}{\sigma}$. We get

$$\begin{aligned}(1-\alpha)ES_\alpha(L^*) &= \int_{\Phi^{-1}(\alpha)}^{\infty} \ell\phi(\ell)\mathrm{d}\ell = \int_{\Phi^{-1}(\alpha)}^{\infty} \ell\frac{1}{\sqrt{2\pi}}e^{-\ell^2/2}\mathrm{d}\ell \\ &= \frac{1}{\sqrt{2\pi}}\int_{\omega(\alpha)}^{\infty} e^{-x}dx, \quad \omega(\alpha) = (\Phi^{-1}(\alpha))^2/2 \\ &= \frac{1}{\sqrt{2\pi}}e^{-\omega(\alpha)} = \phi(\Phi^{-1}(\alpha)).\end{aligned}$$

Hence, $\mathrm{ES}_\alpha = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$

ES for Student's t-distributions

Suppose $L^* = (L - \mu)/\sigma \sim t_\nu$, that is L^* has a "standard" Student's $t-$distribution with ν degrees of freedom ($\nu > 1$)

- t_ν, f_ν are the cdf and the density of standard t, respectively
- $t^{-1} := t_\nu^{-1}(\alpha)$
- $\text{ES}_\alpha(L) = \mu + \sigma \text{ES}_\alpha(L^*)$

$$\text{ES}_\alpha(L^*) = \frac{f_\nu(t^{-1})}{1-\alpha}\left(\frac{\nu + (t^{-1})^2}{\nu - 1}\right)$$

Therefore

$$\text{ES}_\alpha(L) = \mu + \sigma \frac{f_\nu(t^{-1})}{1-\alpha}\left(\frac{\nu + (t^{-1})^2}{\nu - 1}\right)$$

Exercise

ES of a lognormal distribution

Consider an insurance loss random variable $L \sim \log\mathcal{N}(\mu, \sigma^2)$. Show that

$$ES_\alpha(L) = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\Phi(\Phi^{-1}(\alpha) - \sigma)$$

where $\Phi(\cdot)$ is the cdf of a standard normal rv.

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Shortfall-to-quantile ratio

The difference between VaR and ES matters for a heavy-tailed distribution:

Normal $\lim_{\alpha\to 1} \frac{\mathrm{ES}_\alpha}{\mathrm{VaR}_\alpha} = 1$

Student-t $\lim_{\alpha\to 1} \frac{\mathrm{ES}_\alpha}{\mathrm{VaR}_\alpha} = \frac{\nu}{\nu-1} > 1$

(with $\nu = 3$, ES is 50% larger than VaR in the limit for large α).

$\rightarrow$ ES_α is sensitive to the severity of losses exceeding VaR_α.

VaR and ES for stock returns

Example (McNeil et al. (2015)).
Suppose the current value of a position on a particular stock is $V_t = 10000$. Assume X_{t+1} represents daily log-returns on the stock. The (linearized) loss for this portfolio is

$$L^{\Delta}_{t+1} = -V_t X_{t+1}$$

We assume that

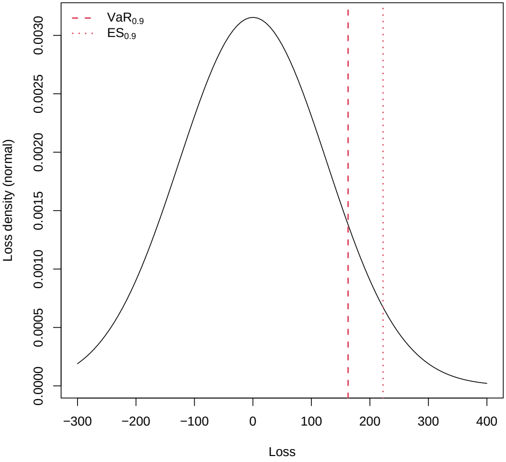
- X_{t+1} has zero mean
- X_{t+1} standard deviation $\sigma_X = 0.2/\sqrt{250}$ (annualized volatility of 20%)

We compare VaR and ES under

1. a normal distribution $\mu = 0, \sigma = V_t \sigma_X$
2. a t-distribution with $\nu = 5$ dof scaled to have standard deviation σ_X

VaR and ES for stock returns

VaR and ES for $\alpha = 0.9, 0.99$, under the normal model (risk measures computed via the R package `qrmtools` [Hofert et al. (2021)])



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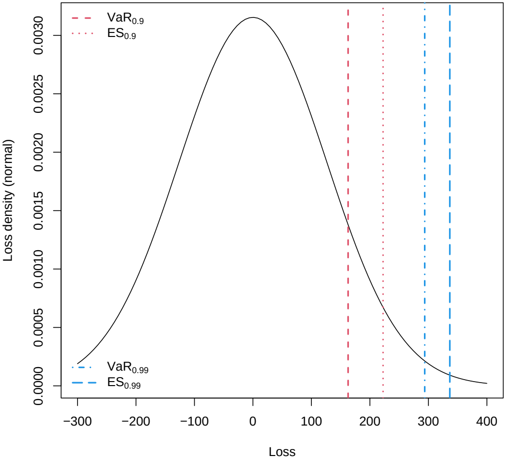

VaR and ES for stock returns

The t distribution is a symmetric distribution with heavy tails
$\rightarrow$ large absolute values are much more probable than in the normal model.

Is the t model riskier than the normal model?

VaR and ES for stock returns

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Is the t model riskier than the normal model?

α	0.9	0.95	0.99	0.995
VaR (normal)	162.10	208.10	294.30	325.80
VaR (t)	144.60	197.40	329.70	395.10
ES (normal)	222.00	260.90	337.10	365.80
ES (t)	225.60	283.20	436.20	514.40

VaR and ES for stock returns: remarks

The normal distribution appears to be at least as risky as the t model using VaR at the 95% or 97.5% level

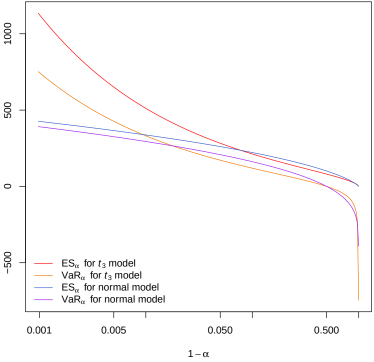
Only for higher α levels (e.g. 99%) the higher risk in the tails of the t model become apparent

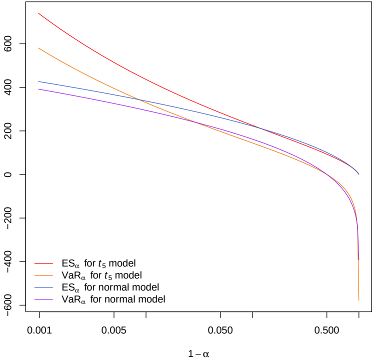
VaR_α (or ES_α) is not always 'riskier' for the t distribution than it is for the normal distribution (only for sufficiently large alpha)

For a heavy-tailed distribution the difference between ES and VaR is more pronounced than for the normal distribution

VaR and ES for stock returns

Comparison between ES/VaR under the normal and t model (the smaller the degrees of freedom the heavier the tails...)





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VaR pros and cons

Some advantages of utilizing VaR include

- possessing a clear interpretation and a relatively simple computation for many distributions with closed-form df
- obtaining VaR of strictly increasing functions by means of the same transformation on the VaR of the original rv
- no additional assumption required

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- no additional assumption required

Major limitations of VaR are

- we can lose (much) more than VaR, depending on the heaviness of the tail of the loss distribution
- VaR **is not a coherent risk measure** implying that diversification benefits may not be fully reflected

Non-subadditivity of VaR

Consider two loss distributions F_{L_1} and F_{L_2} for two portfolios; the overall loss distribution of the merged portfolio $L = L_1 + L_2$ is F_L. It is **not** guaranteed that

$$q_\alpha(F_L) \leq q_\alpha(F_{L_1}) + q_\alpha(F_{L_2})$$

Hence the VaR of the merged portfolio is not necessarily bounded above by the sum of the VaRs of the individual portfolios.

This implies that

- a diversification benefit associated with merging the portfolios is not reflected by VaR
- we cannot be sure that by aggregating VaR numbers for different portfolios we will obtain a bound for the overall risk of the enterprise.

Non-subadditivity: example

Let $L_1, L_2 \sim Exp(1)$ and L_1, L_2 independent $(P(L_1 > x) = P(L_2 > x) = \exp(-x))$. Then, it can be shown that VaR is superadditive, that is,

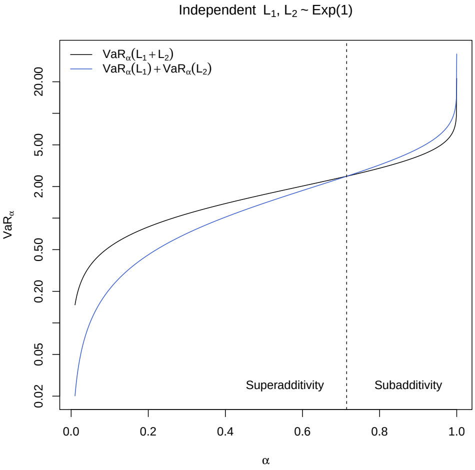
$$\mathrm{VaR}_\alpha(L_1 + L_2) > \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)$$

for $\alpha < 0.71$, and subadditive

$$\mathrm{VaR}_\alpha(L_1 + L_2) \leq \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)$$

otherwise.

Illustration: non-subadditivity of VaR



More on Non-subadditivity of VaR

Formally, consider a risk measure

$$\psi \colon L \to \psi(L)$$

for loss $L \in \mathcal{M}$, a linear space of random variables, which include constants.

Subadditivity $\forall\, L_1, L_2 \in \mathcal{M}: \quad \psi(L_1 + L_2) \leq \psi(L_1) + \psi(L_2)$

Does it matter if a risk measures is subadditive or not?

Subadditivity reflects the idea that risk can be reduced by diversification, and makes decentralization of risk-management systems possible

More on Non-subadditivity of VaR (cont)

Example Consider two trading desks with positions leading to losses L_1 and L_2.
Suppose a risk manager wants to ensure that the risk of the overall loss

$$L = L_1 + L_2$$

is smaller than some number q: if the chosen ψ is subadditive, then find q_1, q_2 such that

$$\psi(L_1) \leq q_1, \quad \psi(L_2) \leq q_2$$

and $q_1 + q_2 \leq q$. Hence

$$\psi(L) = \psi(L_1 + L_2) \leq q_1 + q_2 \leq q.$$

VaR for a portfolio of defaultable bonds

Example: Consider a portfolio of $d = 100$ defaultable corporate bonds:

- assume that defaults of different bonds are independent and default probability is $p = 2\%$.
- $V_0 = 100$ current price of the bonds
- If there is no default, a bond pays in $t + 1$ (one year) an amount of 105

Let L_i be the loss of bond i, then

$$L_i = 100Y_i - 5(1 - Y_i) = 105Y_i - 5$$

where $Y_i = 1$ if bond i defaults in $[t, t + 1]$, and 0 otherwise.

VaR for a portfolio of defaultable bonds (cont)

The losses L_i form a sequence of iid rvs

$$P(L_i = -5) = P(Y_i = 0) = 1 - p = 0.98$$
$$P(L_i = 100) = P(Y_i = 1) = p = 0.02$$

We compare two portfolios, both with current value equal to 10000:

Portfolio A 100 units of bond one (fully concentrated)

$$L_A = 100 L_1$$

Portfolio B one unit of each of the bonds (diversified)

$$L_B = \sum_{i=1}^{100} L_i$$

VaR for a portfolio of defaultable bonds (cont)

We want to compute VaR at a confidence level of 95% for both portfolios:

$$VaR_{0.95}(L_A) = 100\,VaR_{0.95}(L_1);\;\; VaR_{0.95}(L_B) = VaR_{0.95}\left(\sum_{i=1}^{100} 105\,Y_i - 5\right)$$

and $VaR_{0.95}(L_1) = -5$. Moreover,

$$VaR_{0.95}(L_B) = 105\,q_{0.95}(S) - 500$$

where $S = \sum_{i=1}^{100} Y_i \sim Bin(100, 0.02)$. Since, $P(S \leq 5) \approx 0.985$ and $P(S \leq 4) \approx 0.949 < 0.95$, $q_{0.95}(S) = 5$. Hence we get

$$VaR_{0.95}(L_A) = 100(-5) = -500,\;\; VaR_{0.95}(L_B) = 525 - 500 = 25$$

VaR for a portfolio of defaultable bonds (cont)

VaR is not subadditive

$$25 = VaR_{0.95}\left(\sum_{i=1}^{100} L_i\right) > \sum_{i=1}^{100} VaR_{0.95}(L_i) = -500$$

$$VaR_{0.95}(L_B) > VaR_{0.95}(L_A)$$

The risk capital required for portfolio B is higher than for portfolio A:

- an additional risk capital of 25 is required for portfolio B to satisfy a regulator working with VaR at the 95% level
- This contradicts the fact that portfolio B should have a lower VaR being less risky than portfolio A.

VaR for a portfolio of defaultable bonds: Remarks

In the last example, the non-subbaditivity of VaR is caused by the fact that the assets making up the portfolio have very skewed loss distributions. Non-subadditivity of VaR also occurs

- when the underlying rvs are independent but very heavy-tailed
- for dependent losses, when their dependence structure is highly asymmetric

However, it can be shown that VaR is subadditive in the situation where all portfolios can be represented as linear combinations of the same set of underlying elliptically distributed (e.g., normally distributed) risk factors

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Coherent risk measure

Axioms for coherent risk measures

(A1) Monotonicity $\forall\, L_1, L_2 \in \mathcal{M},\ L_1 \leq L_2$ almost surely: $\psi(L_1) \leq \psi(L_2)$

(A2) Translation invariance $\forall\, L \in \mathcal{M}, \ell \in \mathbb{R}$: $\psi(L + \ell) = \psi(L) + \ell$

(A3) Positive homogeneity $\forall\, L \in \mathcal{M}, \lambda > 0$: $\psi(\lambda L) = \lambda \psi(L)$

(A4) Subadditivity $\forall\, L_1, L_2 \in \mathcal{M} :\ \psi(L_1 + L_2) \leq \psi(L_1) + \psi(L_2)$

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(A4) Subadditivity $\forall L_1, L_2 \in \mathcal{M} : \psi(L_1 + L_2) \leq \psi(L_1) + \psi(L_2)$

Remark

- VaR always satisfies (A1)–(A3), but not (A4), in general.
- Expected shortfall is a coherent risk measure (proof omitted)

VaR is coherent in the normal case

The computation of $VaR(L_1 + L_2)$ requires assumptions on the marginals and the dependence between the risks (joint distribution).
If $(L_1, L_2) \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$, then $L_i \sim \mathcal{N}(\mu_i, \sigma_i^2), i = 1, 2$, where

$$\boldsymbol{\mu} = (\mu_1, \mu_2), \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \rho \in [-1, 1] \tag{3}$$

and

$$L_1 + L_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$$

Let $\alpha > 0.5$, then

$$\begin{aligned} VaR_\alpha(L_1 + L_2) &= \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\,\Phi^{-1}(\alpha) \\ &\leq \mu_1 + \mu_2 + \sqrt{(\sigma_1 + \sigma_2)^2}\,\Phi^{-1}(\alpha) \\ &= (\mu_1 + \sigma_1\Phi^{-1}(\alpha)) + (\mu_2 + \sigma_2\Phi^{-1}(\alpha)) \\ &= VaR_\alpha(L_1) + VaR_\alpha(L_2) \end{aligned}$$

Further comments: Choice of VaR parameters

There are two important choices when working with VaR:

Choice of Δt should reflect the time period over which a financial institution is committed to hold its portfolio

- usually one year for measuring the risk in the liability and asset portfolios of an insurer
- Δt should be relatively small to (i) use of the linearized loss operator (ii) assume the composition of the portfolio remains unchanged

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- usually one year for measuring the risk in the liability and asset portfolios of an insurer
- Δt should be relatively small to (i) use of the linearized loss operator (ii) assume the composition of the portfolio remains unchanged

Choice of α can be different according to the specific purpose: for instance, the Basel Committee proposes the use of VaR at the 99% level and $\Delta t = 10$ days for market risk

- in general, capital-adequacy purposes require a high confidence level in order to have a sufficient safety margin

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Variance as a risk measure

The variance of the P&L distribution has been used as a risk measure in finance:

- is a well-understood concept which is easy to use analytically
- in market risk, the *volatility* can be interpreted as a measure of uncertainty

However,

- we have to assume that the second moment of the loss distribution exists
- it makes no distinction between positive and negative deviations from the mean
- variance is a good measure of risk only for distributions which are (approximately) symmetric around the mean, such as the normal distribution or a (finite-variance) Student's t-distribution

Alternative risk measures

Some risk measures have been proposed that are simultaneously coherent and may also consider losses beyond VaR.

In Acerbi and Tasche (2002) are listed five measures of risk that include losses in excess of VaR

- Conditional VaR (CVaR)
- Expected shortfall (ES)
- Tail conditional expectation (TCE)
- Worst conditional expectation (WCE)
- Spectral risk measures

First R lab

We use R compute VaR and ES for the different methods:

- Variance-covariance method (assume that the linearized loss provides a sufficiently accurate approximation and multivariate normal)
- Historical simulation method (using empirically estimated risk measures)
- Monte Carlo simulation method (simulate losses from fitted multivariate t or normal risk-factor changes)

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