Smooth Tangent Vector Fields

Definition

Let $(U, \mathbf{x})$ be a chart of a surface S and $\alpha : I \to S$ a smooth path with $\alpha(I) \subset \mathbf{x}(U)$.
A *smooth tangent vector field* w along α assigns

$$w(t) \in T_{\alpha(t)}S$$

such that

$$w(t) = a(t)\mathbf{x}_u(q) + b(t)\mathbf{x}_v(q), \quad q = \mathbf{x}^{-1}(\alpha(t)),$$

for smooth functions $a, b : I \to \mathbb{R}$.

Example: Velocity Vector Field

Example

Let $\alpha : I \to S$ be smooth with $\alpha(I) \subset \mathbf{x}(U)$. Then $\alpha'(t)$ defines a smooth tangent vector field.
If

$$\alpha(t) = \mathbf{x}(u(t), v(t)),$$

then

$$\alpha'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v.$$

We call $\alpha'(t)$ the *velocity vector field*.

Covariant Derivative

Definition

Let w be a smooth tangent vector field along α. The *covariant derivative* is

$$\frac{Dw}{dt}(t) := \text{projection of } \frac{dw}{dt}(t) \text{ onto } T_{\alpha(t)}S.$$

- $\frac{Dw}{dt}(t) \in T_{\alpha(t)}S$
- Measures change of w *within the surface*

Geodesics

Definition

A curve $\alpha : I \to S$ is *locally geodesic* at s_0 if

$$\frac{D\alpha'}{dt}(s_0) = 0.$$

It is a *geodesic* if this holds for all $t \in I$.

- Generalization of straight lines
- Geodesics locally minimize length
- On a sphere: *great circles*

Geodesics

The distance $d(p, q)$ between two points p and q of S is defined as the *infimum* of the length

$$L(\gamma) = \int_a^b \sqrt{I_{\gamma(t)}(\gamma'(t))}\, dt$$

taken over all continuous, piecewise continuously differentiable curves $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

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Another equivalent way of introducing geodesics on a regular surface, is to define them as the minima of the following *energy functional*

$$E(\gamma) = \frac{1}{2}\int_a^b I_{\gamma(t)}(\gamma'(t))\, dt.$$

Geodesics

Remark

It happens that minimizers of $E(\gamma)$ also minimize $L(\gamma)$, because they turn out to be affinely parameterized, and the inequality is an equality. The usefulness of this approach is that the problem of seeking minimizers of E is a more robust variational problem.

Equations of Geodesics

If $\alpha : I \to \mathbf{x}(U) \subset S$ is a regular curve on a regular surface with a (local) parameterization $\mathbf{x}$, the tangent vector field $\alpha'(t)$ is given by

$$\mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)$$

and

$$\begin{aligned} \frac{d\alpha'(t)}{dt} &= u''\mathbf{x}_u + \mathbf{x}_{uu}(u')^2 + v''\mathbf{x}_v + \mathbf{x}_{vv}(v')^2 + \mathbf{x}_{vu}v'u' + \mathbf{x}_{uv}u'v' \\ &= u''\mathbf{x}_u + \mathbf{x}_{uu}(u')^2 + v''\mathbf{x}_v + \mathbf{x}_{vv}(v')^2 + 2\mathbf{x}_{uv}u'v' \\ &= u''\mathbf{x}_u + (u')^2[\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN] + \\ &+ v''\mathbf{x}_v + (v')^2[\Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN] + \\ &+ 2u'v'[\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN] \end{aligned}$$

since

$$\mathbf{x}_{uu} = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN \quad \mathbf{x}_{uv} = \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN$$
$$\mathbf{x}_{vv} = \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN$$

Geodesics

In other words

$$\begin{aligned}\frac{d\alpha'(t)}{dt} &= [u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2]\mathbf{x}_u + \\ &+ [v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2]\mathbf{x}_v + \\ &+ [(u')^2 e + u'v'f + (v')^2 g]N\end{aligned}$$

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Hence $\alpha'(t)$ is parallel (or $\frac{D\alpha'}{dt} = 0$)

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0$$

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Algebraic Value of the Covariant Derivative

By construction, $\frac{Dw}{dt}(t) \in T_{\alpha(t)}S$. If w is a unit tangent vector field, then $\langle w(t), w(t) \rangle = 1$ for all $t \in I$.

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After differentiating, we obtain

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Let $N(\alpha(t))$ denote the unit normal to the surface at $\alpha(t)$. Since $N(\alpha(t))$ is orthogonal to both $w(t)$ and $\frac{Dw}{dt}(t)$, there exists a smooth function $\lambda : I \to \mathbb{R}$ such that

$$\frac{Dw}{dt}(t) = \lambda(t)\big(N(\alpha(t)) \wedge w(t)\big).$$

We define the *algebraic value* of the covariant derivative by

$$\left[\frac{Dw}{dt}(t)\right] := \lambda(t).$$

Geodesic Curvature

Definition

Let $\alpha : I \to S$ be parametrized by arc length. The *geodesic curvature* is

$$k_g(s) := \left[\frac{D\alpha'}{ds}(s)\right].$$

- $\frac{D\alpha'}{ds}$ is the tangential part of α''
- $k_g(s) = 0$ if α is a geodesic.

Angle Functions

Lemma

If $a, b : I \to \mathbb{R}$ are smooth functions such that $a^2 + b^2 \equiv 1$, then there exists $\varphi : I \to \mathbb{R}$ such that

$$a(t) = \cos\varphi(t), \qquad b(t) = \sin\varphi(t),$$

with

$$\varphi' = ab' - a'b.$$

Difference of Covariant Derivatives

Lemma

Let v, w be smooth unit tangent vector fields along a smooth curve α. If φ is the angle from v to w, then

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt}.$$

Algebraic value in local coordinates

Proposition

Let $\mathbf{x}(u, v)$ *be an orthogonal parametrization of a neighborhood, and let* $w(t)$ *be a differentiable field of unit tangent vectors along the curve* $\mathbf{x}(u(t), v(t))$. *Then*

$$\left[\frac{Dw}{dt}\right] = \left\langle \frac{Dw}{dt}, N \wedge w \right\rangle = \frac{1}{2\sqrt{EG}}\left(G_u \frac{dv}{dt} - E_v \frac{du}{dt}\right) + \frac{d\varphi}{dt},$$

where $\varphi(t)$ *is the oriented angle from* $\mathbf{x}_u$ *to* w.

Proof.

Let

$$e_1 = \frac{\mathbf{x}_u}{\sqrt{E}}, \qquad e_2 = \frac{\mathbf{x}_v}{\sqrt{G}}$$

be the unit vectors tangent to the coordinate curves. Since the parametrization is orthogonal, we have $e_1 \wedge e_2 = N$ $(N \wedge e_1 = e_2)$.

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be the unit vectors tangent to the coordinate curves. Since the parametrization is orthogonal, we have $e_1 \wedge e_2 = N$ $(N \wedge e_1 = e_2)$. By the previous Lemma (with $e_1 = v$)

$$\left\langle \frac{Dw}{dt}, N \wedge w \right\rangle = \frac{d\varphi}{dt} + \left\langle \frac{De_1}{dt}, e_2 \right\rangle. \tag{1.1}$$

□

Proof.

Restricting e_1 to the curve $\mathbf{x}(u(t), v(t))$, we compute

$$\frac{De_1}{dt} = \left\langle \frac{de_1}{dt},\ N \wedge e_1 \right\rangle N \wedge e_1 = \left\langle \frac{de_1}{dt},\ e_2 \right\rangle e_2.$$

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Thus,

$$\left\langle \frac{De_1}{dt},\, e_2 \right\rangle = \langle (e_1)_u, e_2 \rangle \frac{du}{dt} + \langle (e_1)_v, e_2 \rangle \frac{dv}{dt}. \tag{1.2}$$

□

Proof.

Since $F = 0$, we have

$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = -\frac{1}{2} E_v \quad \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2} G_u$$

which implies

$$\langle (e_1)_u, e_2 \rangle = \frac{1}{2\sqrt{EG}} E_v, \quad \langle (e_1)_v, e_2 \rangle = -\frac{1}{2\sqrt{EG}} G_u$$

since

$$(e_1)_u = \mathbf{x}_{uu}/\sqrt{E} + \mathbf{x}_u \cdot (1/\sqrt{E}))_u$$

$$(e_1)_v = \mathbf{x}_{uv}/\sqrt{E} + \mathbf{x}_u \cdot (1/\sqrt{E}))_v$$

and $\langle e_1, e_2 \rangle = 0$.

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and $\langle e_1, e_2 \rangle = 0$.
Substituting into (1.2) yields the desired formula. □

Regular Region

Definition

A region $R \subset S$ is said to be *regular* if R is compact and its boundary ∂R is the finite union of (simple) closed piecewise regular curves which do not intersect each other.

Gauss Bonnet Theorem (Local)

Theorem (Local Gauss–Bonnet))

Let $\mathbf{x}\colon U \to S$ *be an orthogonal parametrization of an oriented surface* S*, where* $U \subset \mathbb{R}^2$ *is homeomorphic to an open disk and* $\mathbf{x}$ *is compatible with the orientation of* S*. Let* $R \subset \mathbf{x}(U)$ *be a regular region of* S*, and let* $\gamma\colon I \to S$ *be such that* $\partial R = \gamma(I)$*.*
If γ *is positively oriented, parametrized by arc length* s*, and if* $\gamma(s_0), \ldots, \gamma(s_k)$ *and* $\phi_0, \ldots, \phi_k$ *are respectively the vertices and external angles of* γ*, then*

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\, ds + \iint_R K\, dA + \sum_{j=0}^{k} \phi_j = 2\pi,$$

where k_g *is the geodesic curvature of the regular arcs of* γ *and* K *is the Gaussian curvature of* S*.*