1) Solve the following indefinite integral:

$$\int 2x\sqrt{1+x^2}\,dx$$

(hint: by substitution and $u = 1 + x^2$

$$u = 1 + x^2$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

Replacing u and dx we get

$$\int 2x\sqrt{u}\frac{du}{2x} = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(1+x^2)^{\frac{3}{2}}$$

2) Find the definite solution of the following differential equation:

$$\frac{dy}{dt} + \frac{2}{t}y - 1 = 0 \wedge y(1) = \frac{5}{3} \wedge t > 0$$

We apply the solution for the general case:

(1) $$y = e^{-\int \frac{2}{t}dt}\left(A + \int e^{\int \frac{2}{t}dt} dt\right)$$

Note that

$\int \frac{2}{t} dt = 2\ln t$

and that:

$e^{-2\ln t} = \frac{1}{t^2}$

$e^{2\ln t} = t^2$

Replacing in (1) we get

$y = \frac{1}{t^2}(A + \int t^2 dt) = \frac{1}{t^2}\left(A + \frac{t^3}{3}\right)$ General solution

Replacing the initial conditions in the general solution we get:

$\frac{5}{3} = \left(A + \frac{1}{3}\right)$

$A = \frac{4}{3}$

$y = \frac{4+t^3}{3t^2}$ Definite solution

3) Write the first order conditions of the following problem and check if they are sufficient conditions.

$$\max_{\{x,y\}} 10x - x^2 + \alpha y \ \ s.t.\ 10x + y = B$$

$$L = 10x - x^2 + \alpha y\ - \lambda(10x + y - B)$$

$$\begin{cases} 10 - 2x - 10\lambda = 0 \\ \alpha - \lambda = 0 \\ 10x + y = B \end{cases}$$

given that constrain is linear, to check sufficiency it is enough to check the concavity of the function $10x - x^2 + \alpha y$.

Its Hessian is: $H = \begin{matrix} -2 & 0 \\ 0 & 0 \end{matrix}$

The two first order principal minors are negative (≤ 0), the second order principal minor is positive (≥ 0). Then the Hessian is semidefinite negative, do the function is concave.
This is enough to state that conditions are sufficient.

4) Consider the following problem:

$$\max_{\{x,y\}} \ y - x^2 \ \ s.t. \ \ x^2 + y^2 \leq 4$$

Write the first order conditions and find the solution

$$L = y - x^2 - \lambda \ \ (x^2 + y^2 - 4)$$

$$\begin{cases} -2x - 2\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ x^2 + y^2 \leq 4, \quad \lambda \geq 0, \quad \lambda\,(x^2 + y^2 - 4) = 0 \end{cases}$$

Case $\lambda = 0$

The second equation becomes $1 = 0$ that is no true. Then in this case we have no solution.

Case $\lambda > 0$

In this case the conditions are

$$\begin{cases} -2x - 2\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ x^2 + y^2 = 4, \end{cases}$$

To satisfy the first equation we need either $\lambda = -1$ or $x = 0$

$\lambda = -1$ does not satisfy the initial assumption of this case $\lambda > 0$

Then we consider $x = 0$

By the third equation we find that either $y = 2$ or $y = -2$.

Replacing $y = -2$ in the second equation and solving by λ we get $\lambda = -\frac{1}{4}$ that does not satisfy the initial assumption of this case $\lambda > 0$

Replacing $y = 2$ in the second equation and solving by λ we get $\lambda = \frac{1}{4}$

Therefore the solution is $x = 0, y = 2$, get $\lambda = \frac{1}{4}$

5) Assume continuous time. At time 1, an agent is endowed with a quantity 10 of resource k. She can consume her endowment between time 1 and time $T = 4$. At time T, resource k has to be non negative. Her utility function is $-\frac{1}{t \cdot c_t}$, where c_t is the quantity of resource k used at time t. Compute the optimal time path for c_t and k_t. (hint: $\frac{dk}{dt} = -c_t$)

$$\max \int_1^4 -\frac{1}{t \cdot c_t}\, dt$$

$$s.t.\ \frac{dk}{dt} = -c_t,\ \ k_4 \geq 0,\ \ k_1 = 10$$

$$H = -\frac{1}{t \cdot c_t} + \lambda_t(-c_t)$$

$$\begin{cases} \frac{1}{t \cdot {c_t}^2} - \lambda_t = 0 \\ \frac{dk}{dt} = -c_t \\ \frac{d\lambda}{dt} = 0 \\ \lambda_4 \geq 0,\ \ k_4 \geq 0,\ \ k_4\lambda_4 = 0 \end{cases}$$

From the third condition we know that $\lambda_t = \lambda$ for every t (it is constant over time)
From the first one we can say that $\lambda_t > 0$, otherwise (when $\lambda_t = 0$) c_t is not defined.
From the last conditions this implies that $k_4 = 0$
From the first equation we have:

$$c_t = \sqrt{\frac{1}{t\,\lambda}}$$

Replacing in the second equation we get:

$$\frac{dk}{dt} = -\sqrt{\frac{1}{t\,\lambda}}$$

The general solution is

$$k_t = -\frac{2t}{\sqrt{\lambda\, t}} + A$$

Using the initial condition $k_1 = 10$ we get the definite solution:

$$k_t = 2\left(5 + \frac{1 - \sqrt{t}}{\sqrt{\lambda}}\right)$$

replacing the terminal condition $k_4 = 0$ and solving by λ we get $\lambda_t = 0.04\ \forall t$
Replacing in the definite solution we find the optimal path for k_t, i.e. $k_t = 20 - 10\sqrt{t}$.
Replacing $\lambda_t = 0.04\ \forall t$ in $c_t = \sqrt{\frac{1}{t\,\lambda}}$ we get the optimal path for c_t, i.e. $c_t = \frac{5}{\sqrt{t}}$.