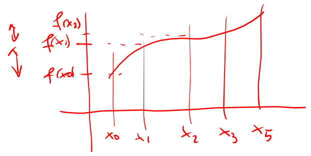
Lesson 4 - 03/03/2026

functions with bounded variation
(bounded variation funct.)

def. let $[a,b] \subseteq \mathbb{R}$
$\Delta = \{a = x_0, x_1, x_2, \dots x_n = b\}$ $(x_0 < x_1 < \dots < x_n)$
is a subdivision of $[a,b]$.

def. $f : [a,b] \to \mathbb{R}$, Δ subdiv.
$$V(f,\Delta) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \quad \text{(the same if } f:[a,b]\to\mathbb{C}\text{)}$$



def. if $\underbrace{\sup_{\Delta} V(f,\Delta)}_{V_a^b(f)} < +\infty$ f is said to be a bounded variation funct.

(total) variation of f

(Rem. let $f : [a,b] \to \mathbb{R}$ continuous
f is rectifiable $\iff$ f is BV)

Elementary properties.

i) $f \in BV([a,b]) \implies f$ is bounded

$$|f(x)| \le |f(a)| + |f(a) - f(x)| \le |f(a)| + \underbrace{|f(a)-f(x)| + |f(x)-f(b)|}_{V(f,\{a,x,b\}) \le V_a^b(f)}$$

$$\|f\|_{L^\infty} \le |f(a)| + V_a^b(f)$$

0) f monotone $\implies f \in BV$ and $V_a^b(f) = |f(b) - f(a)|$

ii) $BV([a,b])$ is vector space
$$V_a^b(f+g) \le V_a^b(f) + V_a^b(g)$$
$$V_a^b(\alpha f) = |\alpha| V_a^b(f)$$

and setting $\|f\|_{BV} = |f(a)| + V_a^b(f)$

BV is a Banach space.

iii) let $f \in BV([a,b])$ let $c \in]a,b[$

then $f_{|[a,c]} \in BV$, $f_{|[c,b]} \in BV$

and $V_a^b(f) = V_a^c(f) + V_c^b(f)$

iv) given $f \in BV([a,b])$ one can consider $\underbrace{x \longmapsto V_a^x(f)}_{\text{this function is increasing}}$

and also $x \mapsto V_a^x(f) - f(x)$ is increasing

in fact consider $x_1, x_2 \in [a,b]$ with $x_1 < x_2$

iv) given $f \in BV([a,b])$ one can consider $x \longmapsto \underbrace{V_a^x(f)}_{}$

this function is increasing

and also $x \mapsto V_a^x(f) - f(x)$ is increasing

in fact consider $x_1, x_2 \in [a,b]$ with $x_1 < x_2$

$$f(x_2) - f(x_1) \leq |f(x_2) - f(x_1)| \leq V_{x_1}^{x_2}(f) = V_a^{x_2}(f) - V_a^{x}(f)$$

in particular $f(x_2) - f(x_1) \leq V_a^{x_2}(f) - V_a^{x_1}(f)$

$$V_a^{x_1}(f) - f(x_1) \leq V_a^{x_2}(f) - f(x_2)$$

<u>Theorem</u> let $f: [a,b] \longrightarrow \mathbb{R}$.

if $f \in BV([a,b])$ then f is a.e. differentiable

proof $f(x) = \underbrace{V_a^x(f)}_{\text{increasing}} - (\underbrace{V_a^x(f) - f}_{\text{increasing}})$

by Lebesgue's th we can conclude

Ex. let $f: [a,b] \to \mathbb{R}$

let $f \in BV([a,b])$

suppose that f is continuous at $x_0 \in]a,b[$

then also $x \mapsto V_a^x(f)$ is continuous at x_0

sol. I prove only that if

$\lim_{x \to x_0^-} f(x) = f(x_0)$

then $\lim_{x \to x_0^-} V_a^x(f) = V_a^{x_0}(f)$

I fix $\varepsilon > 0$ from $\lim_{x \to \bar{x}^-} f(x) = f(\bar{x})$

there exist $\delta > 0$ s.t. $\forall x \in]\bar{x} - \delta, \bar{x}[$

$$|f(x) - f(\bar{x})| < \varepsilon/2$$

I consider $V_a^{\bar{x}}(f)$

$\exists \Delta = \{x_0 = a < x_1 < \dots < x_n = \bar{x}\}$

s.t. $\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| > V_a^{\bar{x}}(f) - \varepsilon/2$

it is not restrictive to have $x_{n-1} \in]\bar{x} - \delta, \bar{x}[$

I remark that $V_a^{x_{n-1}}(f) - \sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})| \geq 0$

$$V_a^{\bar{x}}(f) < \cancel{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|} + \varepsilon/2 + V_a^{x_{n-1}}(f) - \cancel{\sum_{j=1}^{n-1} |f(x_j) - f(x_{j-1})|}$$

I obtain

$$V_a^{\bar{x}}(f) < \underbrace{|f(x_{n-1}) - f(\bar{x})|}_{< \varepsilon/2} + \varepsilon/2 + V_a^{x_{n-1}}(f)$$

$$V_a^{\bar{x}}(f) < \sum_{j=1}^{n} \cancel{|f(x_j) - f(x_{j-1})|} + V_a^{x_{n-1}}(f) - \sum_{j=1}^{n-1} \cancel{|f(x_j) - f(x_{j-1})|} + \varepsilon/2$$

I obtain

$$V_a^{\bar{x}}(f) < \underbrace{|f(x_{n-1}) - f(\bar{x})|}_{< \varepsilon/2} + \varepsilon/2 + V_a^{x_{n-1}}(f)$$

finally $V_a^{\bar{x}}(f) - V_a^{x_{n-1}}(f) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Rightarrow \forall y > x_{n-1} \quad V_a^{y}(f) > V_a^{\bar{x}}(f) - \varepsilon$

$$\lim_{x \to \bar{x}^-} V_a^{x}(f) = V_a^{\bar{x}}(f) \quad !$$

QED

The integral function of $f \in L^1(a,b)$.

Th. let $f \in L^1(a,b)$ $\left(L^1(a,b) = L^1([a,b]) \right)$

consider $F(x) = \int_{[a,x]} f \qquad F : [a,b] \to \mathbb{R}$

Then i) F is uniformly continuous

ii) F is $BV([a,b])$ and $V_a^b(F) = \|f\|_{L^1}$

proof. i). let $(x_n)_n$ a sequence s.t. $x_n \to \bar{x}$

consider $f_n = \chi_{[a,x_n]} \cdot f$

$f_n \to \chi_{[a,\bar{x}]} f$ pointwise (a.e.)

and $|f_n| \le |f|$

dominated conv. $\int f_n \to \int f \cdot \chi_{[a,\bar{x}]}$

$F(x_n) \to F(\bar{x})$

F is continuous. $F : [a,b] \to \mathbb{R} \Rightarrow$ unif. cont.

ii) consider $\Delta = \{ x_0 = a < x_1 < \dots < x_n = b \}$

$$F(x_j) - F(x_{j-1}) = \int_{[x_{j-1}, x_j]} f$$

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \le \sum_{j=1}^{n} \left| \int_{[x_{j-1}, x_j]} f \right|$$

$$\le \sum_{j=1}^{n} \int_{[x_{j-1}, x_j]} |f| = \int_{[a,b]} |f| = \|f\|_{L^1}$$

$\Rightarrow V_a^b(F) \le \|f\|_{L^1} \Rightarrow F \in BV$.

to conclude: it remains to prove $\|f\|_{L^1} \le V_a^b(F)$

to conclude : it remains to prove $\|f\|_{L^1} \le V_a^b(F)$

I need to remember that

if $g \in L^1(a,b)$

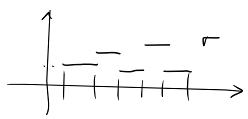
$\exists\, (\sigma_n)_n$ a sequence of simple functions

$\sigma(x) = \sum_{j=1}^{n} \alpha_j \chi_{[x_{j-1},x_j]}(x)$

s.t.

$\sigma_n(x) \to g(x)$ a.e.

and $\|\sigma_n - g\|_{L^1} \xrightarrow{n} 0$



take $f \in L^1(a,b)$

consider $\operatorname{sgn} f(x) = \begin{cases} 1 & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) = 0 \\ -1 & \text{if } f(x) < 0 \end{cases}$

consider $(\sigma_n)_n$ as above , $\sigma_n \to \operatorname{sgn} f$ a.e.
and in $L^1(a,b)$

it is not restrictive to have
$|\alpha_j| \le 1$
$\forall \alpha_j$
s.t.
$\sigma(x) = \sum_j \alpha_j \chi_{[\;]}$ $(*)$

I claim that

$$\int_{[a,b]} |f(x)|\,dx = \lim_n \underbrace{\int_{[a,b]} \sigma_n(x) f(x)\,dx}_{\le V_a^b(F)}$$

why?

dominated convergence

$\sigma_n(x) \cdot f(x) \longrightarrow \operatorname{sgn} f(x) \cdot f(x) = |f(x)|$ a.e.

$|\sigma_n(x) f(x)| \le |f(x)| \quad \forall n \quad (\text{by } *)$

to conclude $\left| \int_{[a,b]} \sigma_n(x) f(x)\,dx \right| \le$?

$$\left| \int \sum_{j=1}^{n} \alpha_j \chi_{[x_{j-1},x_j]}(x) \cdot f(x) \right| \le \sum_j |\alpha_j| \left| \int_{[x_{j-1},x_j]} f(x)\,dx \right|$$

$|\alpha_j| \le 1$

$$\le \sum_{j=1}^{n} \left| \int_{[x_{j-1},x_j]} f(x)\,dx \right| = \sum_j |F(x_j) - F(x_{j-1})|$$

$$\le V_a^b(F)$$

$$\int_a^b |f|\,dx \le V_a^b(F)$$

QED

Corollary. let $f \in L^1(a,b)$

consider $F(x) = \int_a^x f(t)\,dt = \int_{[a,x)} f$

Then F is <u>a.e. differentiable</u>.

Rem. fundamental theorem of calculus (Riemann version)

Rem. fundamental theorem of calculus (Riemann version)

take $f : [a,b] \to \mathbb{R}$,
suppose f is continuous
then f is Riemann-integrable.
so I can consider $F(x) = \int_a^x f(t)\,dt$

then F is differentiable (for every $x \in [a,b]$)
and $F'(x) = f(x)$.

now
take $f \in L^1(a,b)$
then $F(x) = \int_{[a,x]} f$ makes sense

F is almost everywhere differentiable

Q. what about F' ?

Theorem (fund. theorem of calculus, Lebesgue's version)

let $f \in L^1(a,b)$
consider $F(x) = \int_{[a,x]} f$

F is a.e. differentiable and $F'(x) = f(x)$ a.e.

Lemma 1
let $f : [a,b] \to \mathbb{R}$, f increasing
then f' (which exist a.e. by Lebesgue)
$f' \in L^1(a,b)$ and $\int_a^b f'(x)\,dx \le f(b) - f(a)$.

proof. consider

$\left(\frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = \right) \quad g_n(x) = n\left(f(x+\tfrac{1}{n}) - f(x)\right) \quad \left(\begin{matrix}\text{define } f(x) = f(b) \\ \text{for } x > b\end{matrix}\right)$

I know that, for a.e. $x \in [a,b]$

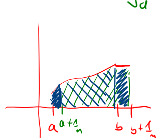
$\lim_n g_n(x) = f'(x)$ and $g_n(x) \geq 0$
(since f is increasing)

I apply Fatou's lemma to $(g_n)_n$

$$\int_a^b \liminf_n g_n(x)\,dx \le \liminf_n \int_a^b g_n(x)\,dx$$

$$\int_a^b \lim_n g_n(x)\,dx$$

$$\int_a^b f'(x)\,dx \le \liminf_n \left(n \int_a^b \left(f(x+\tfrac{1}{n}) - f(x)\right) dx \right)$$



$$\le \liminf_n n \left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(t)\,dt - \int_a^b f(t)\,dt \right)$$

$$\le \liminf_n n \left(\int_b^{b+\frac{1}{n}} f(t)\,dt - \int_a^{a+\frac{1}{n}} f(t)\,dt \right)$$

$$\int_a^b f'(x)\,dx \leq \liminf_n \left(n \int_a^b \left(f(x+\tfrac{1}{n}) - f(x) \right) dx \right)$$

$$\leq \liminf_n n \left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(t)\,dt - \int_a^b f(t)\,dt \right)$$

$$\leq \liminf_n n \left(\int_b^{b+\frac{1}{n}} f(t)\,dt - \int_a^{a+\frac{1}{n}} f(t)\,dt \right)$$

$$\leq \liminf_n \left(\underbrace{n \cdot \tfrac{1}{n} \cdot f(b)}_{f(b)} - n \int_a^{a+\frac{1}{n}} f(t)\,dt \right)$$

$$\leq f(b) + \underbrace{\liminf_n \left(-n \int_a^{a+\frac{1}{n}} f(t)\,dt \right)}_{f(b) - \limsup_n n \int_a^{a+\frac{1}{n}} f(t)\,dt}$$

$$\limsup_n n \int_a^{a+\frac{1}{n}} f(t)\,dt \geq f(a) \Rightarrow -\limsup \leq -f(a)$$

$$\leq f(b) - f(a)$$

QED

lemma let $f \in L^1(a,b)$ suppose that for all $x \in [a,b]$, $\int_{[a,x]} f(t)\,dt = 0$

then $f = 0$ (a.e.)

proof. from the fact that $\int_{[a,x]} f(t)\,dt = 0 \quad \forall x$

I deduce that $\forall\ ]\alpha,\beta[\ \subseteq\ ]a,b[$

$$\int_{[\alpha,\beta]} f(t)\,dt = 0 \qquad \int_{[\alpha,\beta]} \overset{=0}{} = \int_{[a,\beta]} \overset{=0}{} - \int_{[a,\alpha]} \overset{=0}{}$$

I take A open set in $]a,b[$

then $\int_A f(t)\,dt = 0 \qquad \left(A = \bigcup_k]\alpha_k, \beta_k[\right)$

then if K compact in $[a,b]$

$$\int_K f(t)\,dt = 0 \qquad \overset{0}{\overset{\parallel}{\int_{[a,b]}}} = \int_K + \overset{0}{\overset{\parallel}{\int_{[a,b]\setminus K}}}$$

$\underbrace{[a,b]\setminus K}_{\text{open}}$

now suppose $E = \{x \in [a,b] : f(x) > 0\}$
$F = \{x \in [a,b] : f(x) < 0\}$

$E = \bigcup_n E_n \qquad E_n = \{x \in [a,b] : f(x) > \frac{1}{n}\}$

(similarly $F_n = \bigcup_n F_n$ with $F_n, \dots$)

so $\lambda(E_n) = 0 \Rightarrow \lambda(E) = 0$
$\lambda(F_n) = 0 \Rightarrow \lambda(F) = 0$ $\Big\} \Rightarrow f = 0$ a.e.

by contradiction suppose $\exists\, \bar{n}$ s.t. $\lambda(E_{\bar{n}}) > 0$

then take $K \subseteq E_n$

s.t. $\lambda(K) > \frac{\lambda(E_n)}{2}$

compact

proved above

$0 = \int_K f$ · but on K $f(x) > \frac{1}{n}$ and $\lambda(K) > 0$

so that $\int_K f > 0$ ~~[illegible]~~

I prove the theorem

take $f \in L^1(a,b)$

it is not restrictive to have $f \geq 0$

(if not I write $f = f^+ - f^-$ with $f^+, f^- \geq 0$

(I have $F = F^+ - F^-$ $\quad F^+ = \int f^+$, $\quad F^- = \int f^-$ and so on)

Step 1 suppose f bounded

$0 \leq f \leq M$ a.e.

consider $G_n(x) = n\left(F(x+\frac{1}{n}) - F(x)\right)$

(where $F(x) = \int_a^x f$)

I know that $G_n(x) \xrightarrow{n} F'(x)$ a.e.

and $|G_n(x)| \leq n\left(F(x+\frac{1}{n}) - F(x)\right)$

$$\leq n \int_x^{x+\frac{1}{n}} f(t)\,dt \leq n \int_x^{x+\frac{1}{n}} M \leq M$$

$|G_n(x)| \leq M$ dominated convergence

so $\lim_n \int_a^x G_n(t)\,dt = \int_a^x F'(t)\,dt$

but $\int_a^x G_n(t)\,dt = \int_a^x n\left(F(t+\frac{1}{n}) - F(t)\right)dt$

$$= n\int_x^{x+\frac{1}{n}} F(t)\,dt - n\int_a^{a+\frac{1}{n}} F(t)\,dt$$

F is continuous

$$F(x) - \underset{= 0}{F(a)}$$

so $\lim_n \int_a^x G_n(t)\,dt = F(x) = \int_a^x f(t)\,dt$

in conclusion $\int_a^x F'(t)\,dt = \int_a^x f(t)\,dt$

i.e. $\int_a^x \left(F'(t) - f(t)\right)dt = 0$

from the [illegible] lemma $\Rightarrow F' = f$.