Chracterization of BV functions on an interval

Theorem 1 (Helly's selection theorem). *Let (f_n) be a sequence of increasing functions on $[a, b]$. Suppose that there exists $M > 0$ such that, for all $n \in \mathbb{N}$ and for all $x \in [a, b]$,*

$$|f_n(x)| \leq M.$$

Then there exists a subsequence $(f_{n_k})_k$ and there exists an increasing function g on $[a, b]$ such that, for all $x \in [a, b]$,

$$\lim_k f_{n_k}(x) = g(x).$$

Remark 1. *We stress the fact that in the conclusion of Helly's selection theorem the convergence of the subsequence is a convergence in all points of $[a, b]$.*

Proof. Here a proof due to C. Bennewitz (https://math.stackexchange.com/questions/397931/hellys-selection-theorem). The reference for the original proof is: Helly, E. (1912), "Über lineare Funktionaloperationen", Wien. Ber. (in German), 121: 265–297.

The proof is divided in three steps.

- Let $A = \mathbb{Q} \cap [a, b]$. By a diagonal procedure it is possible to extract a subsequence $(f_{n_k})_k$ which is converging at all points of A.

- Define $h(x) = \limsup_k f_{n_k}(x)$. Since the $\limsup_k$ of a sequence g_k of increasing functions defined on $[a, b]$ is an increasing function, the function h is increasing and for all $a \in A$, $h(a) = \lim_k f_{n_k}(a)$. Suppose that h is continuous at x. We claim that $h(x) = \lim_k f_{n_k}(x)$. Let $r, s \in A$ such that $r < x < s$. We have

  $$f_{n_k}(r) - h(s) \leq f_{n_k}(x) - h(x) \leq f_{n_k}(s) - h(r).$$

  We have

  $$\limsup_k (f_{n_k}(x) - h(x)) \leq \limsup_k (f_{n_k}(s) - h(r)) \leq h(s) - h(r)$$

  and

  $$\liminf_k (f_{n_k}(x) - h(x)) \geq \liminf_k (f_{n_k}(r) - h(s)) \leq h(r) - h(s),$$

  so that

  $$h(r)-h(s) \leq \liminf_k (f_{n_k}(x)-h(x)) \leq \limsup_k (f_{n_k}(x)-h(x)) \leq h(s)-h(r).$$

  We know that A is dense and x is point of continuity of h, consequently

  $$\lim_{r\in A, r\to x^-} h(r) = \lim_{s\in A, s\to x^+} h(r) = h(x)$$

  and

  $$\lim_k f_{n_k}(x) = h(x).$$

- The previous point says that $(f_{n_k})_k$ is converging at all points of $[a,\, b]$ apart of a set which is at most countable (i.e. the set in which h is not continuous). A diagonal procedure gives a sub-sub sequence converging at all points of $[a,\, b]$.

□

Corollary 1 (Helly's selection theorem for BV functions)**.** *Let (f_n) be a sequence of $BV([a,\, b])$ functions. Suppose that there exists $M > 0$ such that, for all $n \in \mathbb{N}$,*

$$V_a^b(f_n) \leq M.$$

Then there exists a subsequence $(f_{n_k})_k$ and there exists a $BV([a,\, b])$ function g such that, for all $x \in [a,\, b]$,

$$\lim_k f_{n_k}(x) = g(x).$$

Moreover $V_a^b(g) \leq M$.

Proof. Define $h_n(x) = V_a^x(f_n)$. $(h_n)_n$ is a sequence of uniformly bounded increasing functions on $[a,\, b]$ so that it is possible to apply Helly's selection theorem. Denoting with $(h_{n_k})_k$ the subsequence and with g the limit function, we have

$$\lim_k h_{n_k}(x) = \lim_k V_a^x(f_{n_k}) = g(x)$$

for all $x \in [a,\, b]$. Define then $l_k(x) = V_a^x(f_{n_k}) - f_{n_k}(x)$. The conclusion of the proof is obtained applying also to $(l_k)_k$ the Helly's selection theorem. □

Theorem 2 (Characterisation of BV functions on an interval)**.** *Let $a,\, b \in \mathbb{R}$, with $a < b$. Let $f \in L^1(a,\, b)$. The following items are equivalent.*

i) There exists $M > 0$ such that, for all $h \in\,]0,\, b-a[$,

$$\int_a^{b-h} |f(t+h) - f(t)|\, dt \leq Mh.$$

ii) There exists $\tilde{f} \in BV([a,\, b])$ such that, for almost every $x \in\,]a,\, b[$,

$$\tilde{f}(x) = f(x).$$

Proof. I prove first that *ii)* implies *i)*. Let $\tilde{f} \in BV([a,\, b])$. Let $t,\, t+h \in [a,\, b]$. Then

$$|\tilde{f}(t+h) - \tilde{f}(t)| \leq V_t^{t+h}(\tilde{f}) = V_a^{t+h}(\tilde{f}) - V_a^t(\tilde{f}).$$

Remarking that both the functions

$$t \mapsto |\tilde{f}(t+h) - \tilde{f}(t)| \qquad \text{and} \qquad t \mapsto V_a^{t+h}(\tilde{f}) - V_a^t(\tilde{f})$$

are integrable on the interval $[a,\ b-h]$, we deduce that

$$\begin{aligned}
\int_a^{b-h} |\tilde{f}(t+h)-\tilde{f}(t)|\,dt &\le \int_a^{b-h} (V_a^{t+h}(\tilde{f}) - V_a^t(\tilde{f}))\,dt \\
&\le \int_a^{b-h} V_a^{t+h}(\tilde{f})\,dt - \int_a^{b-h} V_a^t(\tilde{f})\,dt \\
&\le \int_{a+h}^{b} V_a^{t}(\tilde{f})\,dt - \int_a^{b-h} V_a^t(\tilde{f})\,dt \\
&\le \int_{b-h}^{b} V_a^{t}(\tilde{f})\,dt - \int_a^{a+h} V_a^t(\tilde{f})\,dt \\
&\le \int_{b-h}^{b} V_a^{t}(\tilde{f})\,dt \\
&\le hV_a^b(\tilde{f}).
\end{aligned}$$

The conclusion is a consequence of the fact that

$$\int_a^{b-h} |\tilde{f}(t+h)-\tilde{f}(t)|\,dt = \int_a^{b-h} |f(t+h)-f(t)|\,dt.$$

Remark that in *ii*) there is $M = V_a^b(\tilde{f})$.

I prove now that *i*) implies *ii*). Let's define, for $h \in]0,\ b-a[$,

$$g_h(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt, \qquad g_h : [a,\ b-h] \to \mathbb{R}.$$

We have

$$g_h(x) = \frac{1}{h}(F_a(x+h) - F_a(x)),$$

where $F_a : [a,\ b] \to \mathbb{R}$, $F_a(x) = \int_a^x f(t)\,dt$. We know that $F_a(a) = 0$ and for almost all $x \in [a,\ b]$, F_a is differentiable and $F_a'(x) = f(x)$ for almost all $x \in [a,\ b]$. Consequently

$$g_h'(x) = \frac{1}{h}(f(x+h) - f(x)) \text{ a.e. in } [a,\ b-h].$$

Moreover, from the known result on Lebesgue's points for an L^1 function, we have that for almost all $x \in [a,\ b]$,

$$\lim_{h\to 0} g_h(x) = f(x).$$

Define now

$$h_n(x) = g_{\frac{1}{n}}(x),$$

We have

$$h_n'(x) = n(f(x+\frac{1}{n}) - f(x)) \quad \text{a.e. in } [a,\ b-\frac{1}{n}],$$

$$\lim_n h_n(x) = f(x) \quad \text{a.e. in } [a,\ b]$$

and

$$V_a^{b-\frac{1}{n}}(h_n) = \int_a^{b-\frac{1}{n}} |h_n'(x)|\, dx = \int_a^{b-\frac{1}{n}} |n(f(x+\frac{1}{n}) - f(x))|\, dx \le M.$$

We deduce that for all n_0 sufficiently large and for all $n \ge n_0$,

$$h_n \in BV([a,\, b - \frac{1}{n_0}]) \quad \text{and} \quad V_a^{b-\frac{1}{n_0}}(h_n) \le M.$$

Apply now the corollary to Helly's selection theorem to the sequence $(h_n)_{n \ge n_0}$. There exist a subsequence $(h_{n_k})_k$ and a function $g : [a,\, b - \frac{1}{n_0}] \to \mathbb{R}$ such that

$$\lim_k h_{n_k}(x) = g(x) \quad \text{for all } x \in [a,\, b - \frac{1}{n_0}],$$

where it is interesting to remark that $g(x) = f(x)$ for a. e. in $[a,\, b - \frac{1}{n_0}]$.

Consider now a subdivision of $[a,\, b - \frac{1}{n_0}]$,

$$x_0 = a < x_1 < \cdots < x_{L-1} < x_L = b - \frac{1}{n_0},$$

We have

$$\sum_{j=1}^{L} |g(x_j) - g(x_{j-1})| = \lim_k (\sum_{j=1}^{L} |h_{n_k}(x_j) - h_{n_k}(x_{j-1})|)$$

and, for all $k \in \mathbb{N}$,

$$\sum_{j=1}^{L} |h_{n_k}(x_j) - h_{n_k}(x_{j-1})| \le V_a^{b-\frac{1}{n_0}}(h_{n_k}) \le M.$$

As a consequence $g \in BV([a,\, b - \frac{1}{n_0}])$ and $V_a^{b-\frac{1}{n_0}}(g) \le M$. The above remark that $g(x) = f(x)$ for a. e. in $[a,\, b - \frac{1}{n_0}]$ gives the conclusion.

□

Exercize 1. *Let $(g_k)_k$ be a sequence of increasing functions defined on $[a,\, b]$. Suppose that $(g_k)_k$ is uniformly bounded. Let*

$$h(x) = \limsup_k g_k(x).$$

Show that h is a bounded increasing function.

Remember that $h(x) = \limsup_k g_k(x)$ means

- for all $k \in \mathbb{N}$, $g_k(x) \le h(x)$;
- for all $\varepsilon > 0$ and for all $\bar{k} \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \ge \bar{k}$ and $g_k(x) > h(x) - \varepsilon$.

Let $x_1,\, x_2 \in [a,\, b]$ with $x_1 < x_2$. Suppose by contradiction that $h(x_1) > h(x_2)$. Choose $\varepsilon = h(x_1) - h(x_2)$. Then for all $\bar{k} \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \ge \bar{k}$ and $g_k(x_1) > h(x_2)$. But this implies $g_k(x_1) > g_k(x_2)$ and this is impossible.