Regular Region

Definition

A region $R \subset S$ is said to be *regular* if R is compact and its boundary ∂R is the finite union of (simple) closed piecewise regular curves which do not intersect each other.

Theorem (Turning Tangents)

Let $\alpha : [0, l] \to \mathbf{x}(U) \subset S$ be a simple closed, piecewise regular parametrized curve, with vertices α_{s_j} $(j = 0, \ldots, k)$ and external angle ϑ_j at the corresponding vertex α_{s_j}.

Then if $\varphi_j : [s_j, s_{j+1}] \to \mathbb{R}$ is the differentiable function which measures the angle from $\mathbf{x}_u$ to α' we have

$$\sum_{j=0}^{k}[\varphi_j(s_{j+1}) - \varphi_j(s_j)] + \sum_{j=1}^{k}\vartheta_j = \pm 2\pi$$

where the sign depends on the orientation.

Gauss Bonnet Theorem (Local)

Theorem (Local Gauss–Bonnet)

Let $\mathbf{x}\colon U \to S$ be an orthogonal parametrization of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and $\mathbf{x}$ is compatible with the orientation of S. Let $R \subset \mathbf{x}(U)$ be a regular region of S, and let $\gamma\colon I \to S$ be such that $\partial R = \gamma(I)$.

If γ is positively oriented, parametrized by arc length s, and if $\gamma(s_0), \ldots, \gamma(s_k)$ and $\vartheta_0, \ldots, \vartheta_k$ are respectively the vertices and external angles of γ, then

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\, ds + \iint_R K\, dA + \sum_{j=0}^{k} \vartheta_j = 2\pi,$$

where k_g is the geodesic curvature of the regular arcs of γ and K is the Gaussian curvature of S.

Proof.

Let $u = u(s)$, $v = v(s)$ be the expression of the parametrization of γ in the coordinates given by $\mathbf{x}$. Hence,

$$k_g = \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\varphi_j}{ds},$$

where $\varphi_j(s)$ is the differentiable function measuring the positive angle from $\mathbf{x}_u$ to γ' on the interval $[s_j, s_{j+1}]$.

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where $\varphi_j(s)$ is the differentiable function measuring the positive angle from $\mathbf{x}_u$ to γ' on the interval $[s_j, s_{j+1}]$.

Integrating over each interval $[s_j, s_{j+1}]$ and summing, we obtain

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\, ds = \sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \left(\frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du}{ds} \right) ds +$$

$$+ \sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds.$$

The Gauss–Green Theorem states that if $P(u, v)$ and $Q(u, v)$ are differentiable functions on a (simple) region $A \subset \mathbb{R}^2$ whose boundary is parametrized by $u = u(s)$, $v = v(s)$, then

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \left(P\frac{du}{ds} + Q\frac{dv}{ds} \right) ds = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) dudv.$$

Applying the Gauss Green Theorem to $\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\, ds$, we obtain

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\, ds = \iint_{\mathbf{x}^{-1}(R)} \left[\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right] du dv +$$

$$+ \sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds.$$

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$$+ \sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds.$$

Since the parametrization is orthogonal ($F = 0$), we have

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right]$$

and so

$$\iint_{\mathbf{x}^{-1}(R)} \left[\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right] du dv =$$

$$= -\iint_{\mathbf{x}^{-1}(R)} K \sqrt{EG}\, du dv = -\iint_{R} K\, dA.$$

Proof.

From the Theorem of Turning Tangents,

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds = \sum_{j=0}^{k} \big(\varphi_j(s_{j+1}) - \varphi_j(s_j)\big) = \pm 2\pi - \sum_{j=1}^{k} \vartheta_j.$$

where ϑ_j is the external angle at the corresponding vertex α_{s_j}.

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$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds = \sum_{j=0}^{k} \big(\varphi_j(s_{j+1}) - \varphi_j(s_j) \big) = \pm 2\pi - \sum_{j=1}^{k} \vartheta_j.$$

where ϑ_j is the external angle at the corresponding vertex α_{s_j}. Since γ is positively oriented, the sign is positive. Putting everything together yields the desired formula. $\qquad\square$

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Definition

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- three vertices
- three non-vanishing exterior angles

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A *triangulation* of a region $R \subset S$ is a finite collection of triangles covering R with intersections only along edges or vertices.

Triangulations of region of a surface

Proposition

Every regular region of a regular surface admits a triangulation

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Euler Characteristic

Definition

Let V, E, F be the number of vertices, edges, and faces of a triangulation of R. Then *Euler characteristic* of R is

$$\chi(R) = V - E + F.$$

Lemma

$\chi(R)$ *is independent of the triangulation.*

Connected sum

Definition

Let S_1 and S_2 be two compact connected surfaces. The connected sum of S_1 and S_2, denoted by $S_1 \# S_2$, is the surface defined by removing an open disk from both surfaces and identifying the resulting boundaries via a homeomorphism.

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Lemma
Let S_1 and S_2 be two compact connected surfaces. Then the Euler characteristic satisfies

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

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If S is a compact, connected and orientable surface, then S is homeomorphic either to a sphere or to a connected sum of tori.

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The *genus* of S is $g = \dfrac{2 - \chi(S)}{2}$.

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Definition

The *genus* of S is $g = \dfrac{2 - \chi(S)}{2}$.

- Sphere: $g = 0$
- Torus: $g = 1$
- Each added handle increases g by 1

Theorema Elegantissimum

Theorem (Global Gauss–Bonnet Theorem)

Let $R \subset S$ be a regular region of an oriented surface, and let $C_1, \ldots, C_n$ be the closed, simple, piecewise regular curves forming the boundary ∂R. Suppose that each C_j is positively oriented, and let $\vartheta_1, \ldots, \vartheta_p$ be the external angles of all the curves $C_1, \ldots, C_n$. Then

$$\sum_{j=1}^{n} \int_{C_j} k_g(s)\, ds + \iint_R K\, dA + \sum_{j=1}^{p} \vartheta_j = 2\pi\chi(R),$$

where s denotes arc length along C_j, the integral over C_j is the sum over its regular arcs, and

$$\chi(R) = F - E + V$$

is the Euler–Poincaré characteristic of R associated with a triangulation having F faces, E edges, and V vertices.

Proof.

By a theorem in Topology, there exists a triangulation $\mathcal{J}$ of R such that each triangle is contained in a coordinate neighborhood of an orthogonal parametrization compatible with the orientation of S.

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Applying the Local Gauss–Bonnet Theorem to each triangle and summing the results, the contributions along interior edges cancel due to opposite orientations. Hence,

$$\sum_j \int_{C_j} k_g(s)\,ds + \iint_R K\,dA + \sum_{j,k} \vartheta_{j,k} = 2\pi F,$$

where F is the number of triangles in $\mathcal{J}$ and $\vartheta_{j,1}, \vartheta_{j,2}, \vartheta_{j,3}$ are the external angles of the triangle T_j.

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By a theorem in Topology, there exists a triangulation $\mathcal{J}$ of R such that each triangle is contained in a coordinate neighborhood of an orthogonal parametrization compatible with the orientation of S.

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$$\sum_j \int_{C_j} k_g(s)\, ds + \iint_R K\, dA + \sum_{j,k} \vartheta_{j,k} = 2\pi F,$$

where F is the number of triangles in $\mathcal{J}$ and $\vartheta_{j,1}, \vartheta_{j,2}, \vartheta_{j,3}$ are the external angles of the triangle T_j. Let $\psi_{j,k}$ denote the interior angles of T_j, namely

$$\psi_{j,k} = \pi - \vartheta_{j,k};$$

therefore

$$\sum_{j,k} \vartheta_{j,k} = 3\pi F - \sum_{j,k} \psi_{j,k}.$$

Proof.

Let V_i count the number of internal vertices of the triangulation and let V_e count the number of vertices on the boundary of the region and use similar notation for the edges. Using the combinatorial relations of the triangulation

$$V = V_e + V_i \quad E = E_e + E_i$$

$$V_e = E_e i \quad 3F = 2E_i + E_e$$

since the sum of the internal angles which have an internal common vertex is 2π whereas the sum of the internal angles which have an external common vertex but not a vertex of the region is π, one has

$$\sum_{j=1}^{n} \int_{C_j} k_g(s)\, ds + \iint_R K\, dA + \sum_{j=1}^{p} \varphi_j = 2\pi(F - E + V),$$

which completes the proof. □