The Euler Characteristic

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Abstract

The paper serves as an exposition of the Euler characteristic. We begin with a brief historical overview and continue by introducing n-simplices and Δ-complexes, illustrating the latter by giving the torus a Δ-complex structure. Using this framework, we define the Euler characteristic and give an intuitive argument for its well-definedness. As an application, we consider the connected sum of surfaces and use it to show that the Euler characteristic of the g-torus is $2 - 2g$. Finally, we highlight some examples where the Euler characteristic plays an essential role.

1 Introduction – Historical Overview

The Euler characteristic is among the most fundamental invariants in mathematics. Since its introduction, it has connected a wide variety of mathematical areas, including for example topology, geometry, and combinatorics.

While studying convex polyhedra, Euler observed that in any such polyhedron, the number of vertices v plus the number of faces f differ only by two from the number of edges e, i.e. $v - e + f = 2$. Although Euler did not provide a proof himself, he published this conjecture in 1758. However, it is believed that this fact was already known by Descartes a century prior. Since then, the Euler characteristic has been studied by many mathematicians in various areas. To mention a few results: the Euler-Poincaré formula relates the Euler characteristic to Betti numbers, which are central in algebraic topology and differential geometry; Betti numbers and the Euler characteristic were shown to be homotopy invariant; in 1976, the Atiyah-ℓ_2-Euler-Poincaré formula was established, which links the ℓ_2-Euler-Poincaré characteristic to the ordinary Euler-Poincaré characteristic; and in 1995, Gromov showed a relation between the ℓ_2-Euler-Poincaré characteristic and the fundamental group in the case of Kähler surfaces. More details on these milestones, including clarifications on terminology, can be found in [4, pp. 177-188].

In what follows, we will assume that the reader is familiar with basic notions of topology, real analysis, and linear algebra.

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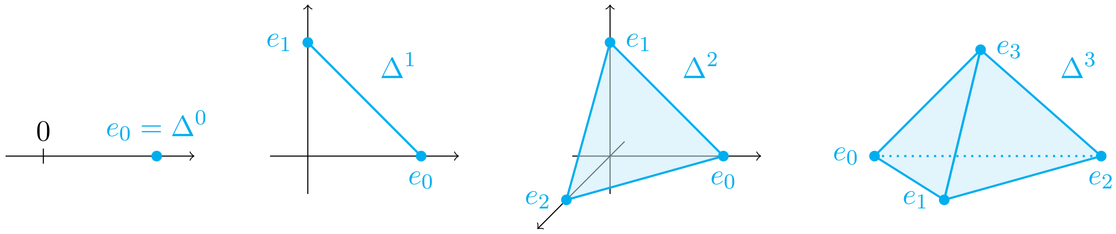

Figure 1: From left to right we have visualisations of the standard simplices Δ^0, Δ^1, Δ^2, and Δ^3.

2 Definition

Depending on the mathematical structure under consideration, there are many different ways to define the Euler characteristic, some of which are more general than others. In this section, we will introduce the Euler characteristic using Δ-complexes. Note that this approach is a special case of the definition as stated in [5, p. 146] using CW complexes. A more general definition, which we will not discuss in this exposition, arises from the previously mentioned Betti numbers, which are defined as the ranks of the homology groups of a topological space. For more details on homology groups, Betti numbers, and the corresponding definition of the Euler characteristic, see [1, pp. 168–172, 215].

First, let us introduce the notion of a Δ-complex, following [5]. Intuitively, one can think of a Δ-complex as a collection of n-triangles glued together along their boundaries. These n-triangles are called n-simplices and are defined as follows:

Definition 2.1 (n-Simplex, [5, p. 103]) *Let $m \geq n \geq 0$ be non-negative integers. An **n-simplex** in $\mathbb{R}^m$ is the convex hull of a set $V \subseteq \mathbb{R}^m$ of $n+1$ points that are not contained in an affine $(n-1)$-dimensional subspace of $\mathbb{R}^m$. The points in V are called vertices. An n-simplex is an **ordered** n**-simplex** if there exists an ordering $v_0 \leq \cdots \leq v_n$ on the vertices in V. We will denote it by $[v_0, \ldots, v_n]$.*

*The **standard** n**-simplex** Δ^n is the convex hull of the standard basis $\{e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$, as a set*

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, \text{ and } x_i \geq 0 \text{ for all } i = 0, \ldots, n\}.$$

The ordered standard n-simplex is the ordered simplex $[e_0, \ldots, e_n]$.

Illustrations of the standard simplices Δ^0, Δ^1, Δ^2, and Δ^3 are given in Figure 1.

Given these n-simplices, we may define a Δ-complex. The idea is to inductively map n-simplices to a topological space X via continuous maps such that their interiors are embedded. These maps determine how the simplices are glued together along their boundaries in the topological space. In other words, the image of the boundary of each n-simplex is the union of $(n-1)$-simplices. These $(n-1)$-simplices are called the **faces** of the n-simplex for $n \geq 1$. For example, the boundary of the standard 2-simplex $[e_0, e_1, e_2]$ consists of the three 1-faces $[e_0, e_1], [e_0, e_2]$, and $[e_1, e_2]$.

Definition 2.2 (Δ-Complex, [5, p. 103]) *Let X be a topological space, $\mathcal{A}$ an arbitrary set, and for each $\alpha \in \mathcal{A}$ let $n(\alpha) \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. A Δ-complex structure on X is a collection of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$, where $\Delta^{n(\alpha)}$ is an ordered $n(\alpha)$-simplex for all $\alpha \in \mathcal{A}$, such that*

1. *for every* $\alpha \in \mathcal{A}$ *the restriction* $\sigma_\alpha|_{int(\Delta^{n(\alpha)})}$ *to the interior of* $\Delta^{n(\alpha)}$ *is injective, and for all* $x \in X$ *there exists a unique* $\alpha \in \mathcal{A}$ *with* $x \in \sigma_\alpha(int(\Delta^{n(\alpha)}))$*,*
2. *for every* $\alpha \in \mathcal{A}$ *and every face of* $\Delta^{n(\alpha)}$ *the restriction of* σ_α *to the face coincides with* σ_β *for some* $\beta \in \mathcal{A}$ *such that* $n(\beta) = n(\alpha) - 1$*, and*
3. *a subset* $U \subseteq X$ *is open if and only if* $\sigma_\alpha^{-1}(U)$ *is open in* $\Delta^{n(\alpha)}$ *for every* $\alpha \in \mathcal{A}$*.*

Strictly speaking, we have not defined what an n-simplex in a general topological space X is. However, in the context of a Δ-complex structure on X, we will refer to each map σ_α as an $n(\alpha)$-simplex of X.

Moreover, if we consider an ordered n-simplex $[v_0, \dots, v_n]$, the order of the vertices $v_0, \dots, v_n$ imposes an order on each of the $(n-1)$-faces, which can be interpreted as an orientation of the faces. For instance, in the standard 2-simplex Δ^2, the face $[e_0, e_1]$ can be oriented from e_0 to e_1, the face $[e_0, e_2]$ from e_0 to e_2, and the face $[e_1, e_2]$ from e_1 to e_2. In particular, since a Δ-complex is a collection of standard simplices, these orientations must be respected while gluing, which is implicitly enforced in the second condition in Definition 2.2.

Example 2.3 (Torus, [5, p. 102]) The torus T can be given a Δ-complex structure by subdividing the square into two 2-simplices along a diagonal. This Δ-complex structure consists of two 2-simplices U and V, three 1-simplices a, b, and c, and one 0-simplex v. This indeed defines a Δ-complex structure on T: the interior of each simplex is mapped injectively onto the torus, the orientation of the faces of U and V is respected, and the orientation of the faces of a, b, and c is trivially satisfied since there is only one 0-simplex.

For an illustration of this Δ-complex structure on the torus, please refer to Figure 2.

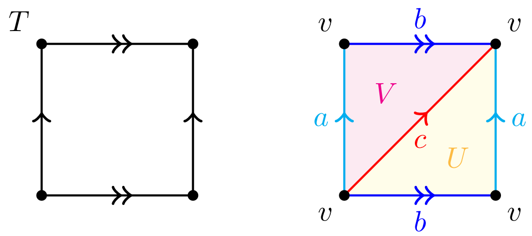

Figure 2: On the left we have the torus T and on the right a possible Δ-complex structure as described in Example 2.3, consisting of two 2-simplices U, V, three 1-simplices a, b, c, and one 0-simplex v.

With these definitions in place, we now define the Euler characteristic for Δ-complexes [5].

Definition 2.4 (Euler Characteristic, [5, p. 146]) *Let* X *be a topological space with a finite* Δ*-complex structure, that is, a* Δ*-complex structure with only finitely many simplices. Then we define the Euler characteristic of* X *by*

$$\chi(X) := \sum_{n \geq 0} (-1)^n \delta_n,$$

where δ_n *denotes the number of* n*-simplices in the* Δ*-complex structure for all* $n \geq 0$*.*

Using the Δ-complex structure of the torus as described in Example 2.3, we see that the Euler characteristic of the torus is $\chi(T) = 0$.

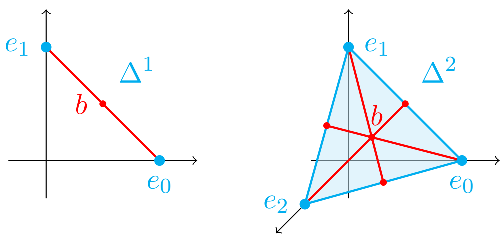

Figure 3: The barycentric subdivision of the standard simplices Δ^1 and Δ^2, where b denotes the respective barycentres.

Remark 2.5 A priori, it is not clear that the Euler characteristic is well-defined. In other words, it is not immediately apparent why the Euler characteristic does not depend on the chosen Δ-complex structure on X. Since a full proof would be too large an excursion at this point, we will only provide the idea: Suppose that we have two distinct finite Δ-complex structures A and B on the topological space X. Using a process called barycentric subdivision, we refine the simplices of both structures until they coincide. The barycentric subdivisions of the standard simplices Δ^1 and Δ^2 are illustrated in Figure 3. Then the key idea is that barycentric subdivision induces chain homotopy equivalence, which leaves the Euler characteristic unchanged. For more details on the well-definedness of the Euler characteristic, see [5, pp. 119–124, 146].

3 Application: Connected Sum & g-Torus

In this section, let us restrict our attention to compact connected surfaces. A **surface** is a two-dimensional real manifold; readers not familiar with the notion of manifold may think of a smooth two-dimensional submanifold embedded in some Euclidean space $\mathbb{R}^n$. In particular, having a Δ-complex structure on a surface is equivalent to specifying a triangulation on the surface, that is, a decomposition of the surface into vertices, edges, and triangular faces whose intersection pattern satisfies the conditions in Definition 2.2. The collection of vertices and edges resembles a graph embedded in the surface. Moreover, note that this perspective also gives an intuitive explanation for why the Euler characteristic of a compact connected surface is well-defined. For this, choose any graph on the surface such that at any intersection of two edges there is a vertex, no edge ends without a vertex, and the complement of the graph is a disjoint union of open disks. Then refine the graph by adding vertices and edges so that each face of the resulting graph is a triangle. This gives a triangulation of the surface, which in turn naturally defines a Δ-complex structure.

The aim in this section is to compute the Euler characteristic of an orientable surface of genus g. Intuitively, one can think of a genus g surface as a compact surface with g holes. For example, the sphere has a genus 0 and the torus has genus 1. In other words, an orientable genus g surface can be obtained by "gluing" g tori to each other. This "gluing" of two surfaces is called the connected sum of the surfaces.

Definition 3.1 (Connected Sum, [3, Definition 2.4.2]) *Let S_1 and S_2 be two compact connected surfaces. The **connected sum** of S_1 and S_2, denoted by $S_1 \# S_2$, is the surface defined by removing an open disk from both surfaces and identifying the resulting boundaries via a homeomorphism.*

It can be shown that the connected sum is independent of the choice of disks: If D_1, D_1' are open disks on S_1, and D_2, D_2' on S_2, then there exist homeomorphisms between $S_1 \setminus D_1$ and

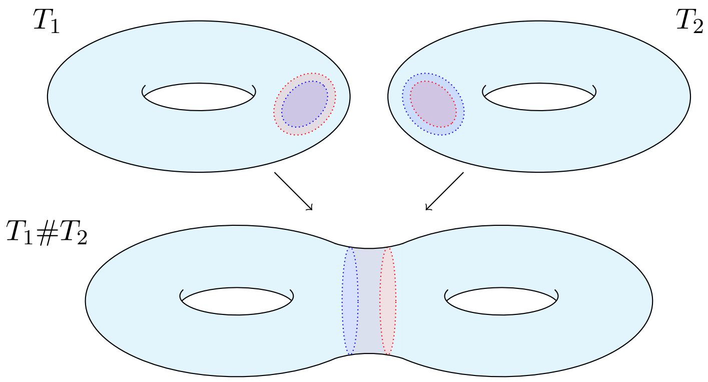

Figure 4: Demonstration of the connected sum $T_1 \# T_2$ of the two tori T_1 and T_2 such that the resulting topological space is a smooth surface by identifying two annular regions.

$S_1 \setminus D_1'$, and between $S_2 \setminus D_2$ and $S_2 \setminus D_2'$, respectively. These homeomorphisms ensure that the connected sum is indeed independent of the choice of disks. Similarly, it is independent of the choice of homeomorphism in Definition 3.1, and hence the connected sum is well-defined.

Moreover, to ensure smoothness of the connected sum at the identified boundaries, we could consider two nested disks on each surface, remove the smaller one, and identify the two resulting annular regions. This construction gives a smooth connected sum, as illustrated in Figure 4. However, for simplicity, we will work with the connected sum as stated in Definition 3.1, i.e. we only identify the boundaries of one removed disk each.

Lemma 3.2 (Euler characteristic & connected sum, [3, Exercise 2.4.7]) *Let S_1 and S_2 be two compact connected surfaces. Then the Euler characteristic satisfies*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proof: Fix Δ-complex structures on both surfaces. As the connected sum is independent of the choice of open disk, we may assume without loss of generality that these disks are 2-simplices of the Δ-complex structures. By removing those 2-simplices and identifying the resulting boundaries, which consist of 1-simplices, we lose two 2-simplices, three 1-simplices, and three 0-simplices. So, if $\chi(S_1) = V_1 - E_1 + F_1$ and $\chi(S_2) = V_2 - E_2 + F_2$, then we obtain

$$\chi(S_1 \# S_2) = V_1 - E_1 + F_1 + (V_2 - 3) - (E_2 - 3) + (F_2 - 2) = \chi(S_1) + \chi(S_2) - 2. \qquad \square$$

Finally, with this, we can now determine the Euler characteristic of the g-torus for $g \geq 1$, i.e. the connected sum $T \# \dots \# T$ of g tori, which we denote by $T^{\# g}$.

Proposition 3.3 (g-Torus, [3, Exercise 2.4.8]) *If $T^{\# g}$ is the g-torus for $g \geq 1$, then*

$$\chi(T^{\# g}) = 2 - 2g.$$

Proof: We will prove the statement by induction on g. First we consider the case $g = 1$: In Example 2.3 we have constructed a Δ-complex structure on the torus T with two 2-simplices, three 1-simplices, and one 0-simplex. Therefore, we have $\chi(T) = 1 - 3 + 2 = 0$.

Next, for $g > 1$ suppose we have already shown that $\chi(T^{\#(g-1)}) = 2 - 2(g-1)$. Then, since $T^{\# g}$ is compact and connected by the definition of connected sum, we can use Lemma 3.2 and the fact that $\chi(T) = 0$ to obtain

$$\chi(T^{\# g}) = \chi(T^{\#(g-1)} \# T) = \chi(T^{\#(g-1)}) + \chi(T) - 2 = 2 - 2(g-1) - 2 = 2 - 2g. \qquad \square$$

In fact, together with orientability, the Euler characteristic is an essential invariant for the classification of compact surfaces. The classification of compact surfaces states that every compact connected surface is homeomorphic to the sphere, the connected sum of g tori, or the connected sum of m projective planes. For more details on the classification of compact surfaces, see [6, Chapter 12].

4 Outlook

The Euler characteristic is not only a beautiful invariant in its own right, but it also plays a crucial role in many important results across various areas of mathematics. In the previous section, we mentioned the classification of compact surfaces in algebraic topology [6]. Another application in the field of differential geometry is the Gauss-Bonnet theorem, which links the integral of the Gaussian curvature of a compact surface to its Euler characteristic [2, pp. 267–279]. In algebraic geometry, on the other hand, the Riemann-Hurwitz formula describes how the Euler characteristics of compact Riemann surfaces are related under holomorphic, possibly branched, coverings [3, Theorem 4.4.1].

The above instances (and many others in which the Euler characteristic appears) underline its beauty and significance in mathematics. It connects different mathematical areas in sometimes unexpected ways and opens new doors for further exploration and collaboration.

References

[1] G. E. Bredon. *Topology and Geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer, New York, NY, 1 edition, 1993. Graduate Texts in Mathematics, vol. 139.

[2] M. P. d. Carmo. *Differential geometry of curves & surfaces*. Dover Publications, Inc., Mineola, New York, revised and updated second edition. edition, 2016.

[3] R. Cavalieri and E. Miles. *Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory*. London Mathematical Society Student Texts. Cambridge University Press, 2016.

[4] B. Eckmann. *Mathematical survey lectures 1943-2004*. Springer, Berlin, 2006.

[5] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[6] J. R. Munkres. *Topology*. Prentice-Hall, Upper Saddle River, 2nd ed. edition, 2000.