Chapter Outline

Finite-Amplitude Waves

- Finite-Amplitude Wave Theory Formulation
- Stokes Waves
- Cnoidal Waves
- Solitary Waves
- Wave Theory Application

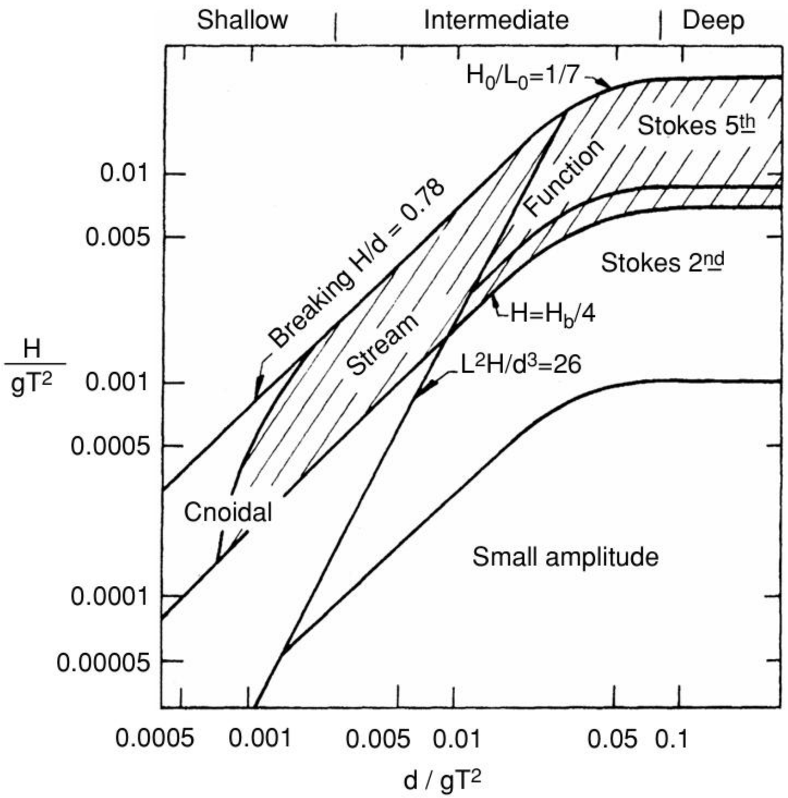
Prof. Marta Cianferra e-mail: marta.cianferra@dia.units.it

Range of applications

Limitations of the small-amplitude theory in shallow water and for high waves in deep water suggest a need to consider nonlinear or finite-amplitude wave theories for some engineering applications. This section presents an over view of selected aspects of the more useful finite-amplitude wave theories, as well as their application and the improved understanding of wave characteristics that they provide.

Remind the two surface boundary conditions for the Laplace equation, had to be linearized and then applied at the still water level. This requires that H/d and H/L be small compared to unity (i.e. small-amplitude theory).

Consequently, the small-amplitude wave theory can be applied over the complete range of relative water depths (d/L), but it is limited to waves of relatively small amplitude relative to the water depth and wave length.



Prof. Marta Cianferra e-mail: marta.cianferra@dia.units.it

Finite-Amplitude Wave Theory Formulation

For numerical theories a computer solution of the numerical equations yields tabulated values of the desired wave characteristics, such as:

- the surface profile
- particle velocity and acceleration
- dynamic pressure
- energy and momentum flux

as a function of selected values of wave height H and period T and water depth d.

On the other hand, the analytical theories produce specific equations for the various wave characteristics which are given in terms of the wave height and period and the water depth.

Both numerical and analytical theories are not complete solutions of the wave boundary value problem, but infinite series solutions that must be truncated at some point (e.g., truncation of a series after the third term yields a third-order solution).

- ❑ The Stokes theory for deep water waves and the cnoidal and solitary theories for shallow water waves are useful analytical theories.
- ❑ Dean's stream function numerical wave theory is a commonly used numerical theory applicable to finite-amplitude waves throughout the range of relative water depths.

Prof. Marta Cianferra e-mail: marta.cianferra@dia.units.it

Stokes Waves

Considering the non-linearity of the boundary condition (i.e. free surface dynamics), Stokes formulated the solution as a power series of H/L

$$\eta = (H/L)\ \eta_1 + (H/L)^2\ \eta_2 + (H/L)^3\ \eta_3 + \ \dots$$
$$\phi = (H/L)\ \phi_1 + (H/L)^2\ \phi_2 + (H/L)^3\ \phi_3 + \ \dots$$

Linear relation — 2nd order — 3rd order

Various higher order approximations to the Stokes theory have been developed. For example, Skjelbreia (1959) proposed a third-order theory, Skjelbreia and Hendrickson (1961) worked on a fifth-order theory, and Schwartz (1974) for much higher order solutions based on calculations using a powerful computer.

Second-order velocity potential and surface function are

$$\Phi = \frac{gH}{2\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx-\sigma t) + \frac{3\pi CH}{16}\left(\frac{H}{L}\right)\frac{\cosh 2k(d+z)}{\sinh^4(kd)}\sin 2(kx-\sigma t)$$

$$\eta = \frac{H}{2}\cos(kx-\sigma t) + \frac{\pi H}{8}\left(\frac{H}{L}\right)\frac{\cosh kd(2+\cosh 2kd)}{\sinh^3 kd}\cos 2(kx-\sigma t)$$

2nd order term

Steepness

Prof. Marta Cianferra e-mail: marta.cianferra@dia.units.it

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Linear relation

The effect of the second-order term having twice the frequency of the small-amplitude term is that the two components of surface amplitude reinforce (i.e., are in phase) each other at the wave crest and oppose each other at the wave trough.

This yields a surface profile vertical asymmetry (more peaked wave crest and flatter wave trough than a cosine profile given by the small-amplitude theory) that grows as the wave steepness increases.

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2nd order term

Steepness

Stokes Waves

Second-order celerity expression remains the same as in small-amplitude theory, while the 3rd order expansion gives

$$C^2 = \frac{g}{k} \tanh kd \left[1 + \left(\frac{\pi H}{L} \right)^2 \left(\frac{9 + 8\cosh^4 (kd) - 8\cosh^2 (kd)}{8\sinh^4 (kd)} \right) \right]$$

Linear relation

Thus, to the third order, wave celerity is amplitude as well as period dispersive: for the same wave period higher waves travel faster than lower waves.

For the limiting steepness in deep water ($H_0/L_0 = 1/7$) the third-order theory yields a wave celerity that is about 10% greater than the celerity given by the small amplitude theory → means that the wavelength would also be 10% larger (since L=C/T).

Stokes Waves

→ The particle velocity and acceleration are increased under the wave crest and diminished under the wave trough. Again, these asymmetries increase as the wave steepness increases.

Since the horizontal component of particle velocity is maximum at the wave crest and trough (and zero at the still water positions), this crest/trough asymmetry in velocity causes particle orbits that are not closed and results in a small drift of the water particles in the direction of wave propagation.

$$u = \frac{\pi H}{T}\frac{\cosh k(d+z)}{\sinh kd}\cos(kx-\sigma t) + \frac{3(\pi H)^2}{4TL}\frac{\cosh 2k(d+z)}{\sinh^4(kd)}\cos 2(kx-\sigma t)$$

$$w = \frac{\pi H}{T}\frac{\sinh k(d+z)}{\sinh kd}\sin(kx-\sigma t) + \frac{3(\pi H)^2}{4TL}\frac{\sinh 2k(d+z)}{\sinh^4(kd)}\sin 2(kx-\sigma t)$$

Second-order particle displacement equations are

$$\zeta = -\frac{H}{2}\frac{\cosh k(d+z)}{\sinh kd}\sin(kx-\sigma t) + \frac{\pi H^2}{8L\sinh^2(kd)}\left(1-\frac{3\cosh 2k(d+z)}{2\sinh^2(kd)}\right)\sin 2(kx-\sigma t) + \frac{\pi H^2}{4L}\frac{\cosh 2k(d+z)}{\sinh^2(kd)}\sigma t$$

$$\varepsilon = \frac{H}{2}\frac{\sinh k(d+z)}{\sinh kd}\cos(kx-\sigma t) + \frac{3\pi H^2}{16L}\frac{\sinh 2k(d+z)}{\sinh^4(kd)}\cos 2(kx-\sigma t)$$

Note that the last term is not periodic but continually increases with time, indicating a net forward transport of water particles as the wave propagates.

Stokes Waves

If we divide the last term by time we have the second-order equation for the mass transport velocity.

$$\bar{u} = \frac{\pi^2 H^2}{2TL} \frac{\cosh 2k(d+z)}{\sinh^2 (kd)}$$

Stokes drift

Since the surface particle velocity at the crest of a wave in deep water is $\pi H/T$ to the first order, $\bar{u}$ indicates that the surface mass transport velocity is of the order of the crest particle velocity times the wave steepness and thus generally much smaller than the crest particle velocity.

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Cnoidal Waves

The applicability of Stokes theory diminishes as a wave propagates across decreasing intermediate/shallow water depths.

Keulegan (1950) recommended a range for Stokes theory application extending from deep water to the point where the relative depth d/L is approximately 0.1. However, the actual Stokes theory cutoff point in intermediate water depths depends also on the wave steepness H/L. For steeper waves, the higher order terms in the Stokes theory begin to unrealistically distort results

→ Cnoidal wave theory (and in very shallow water, solitary wave theory) are the analytical theories most commonly used for shallower water. Cnoidal wave theory is based on equations developed by Korteweg and de Vries (1895).
The resulting equations contain Jacobian elliptical functions, commonly designated cn, so the name cnoidal is used to designate this wave theory.

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Cnoidal Waves

Some of the basic wave characteristics from cnoidal theory can be presented by diagrams, such as:

1) surface profile
2) wave celerity

Diagrams that are based on two parameters: **k^2** and **Ur**.

k^2 is a function of the water depth, the wave length, and max/min of η. It varies from 0 for the small-amplitude limit to 1.0 for the solitary wave limit.

Ur (Ursell number) is a dimensionless parameter given as L^2H/d^3 that is also useful for defining the range of application for various wave theories.

Small amplitude	**$k^2 \to 0$**	Almost sinusoidal wave
Intermediate	**$0 < k^2 < 1$**	Cnoidal wave
Limit	**$k^2 \to 1$**	Solitary wave

$$U_r = L^2 H / d^3$$

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Cnoidal Waves

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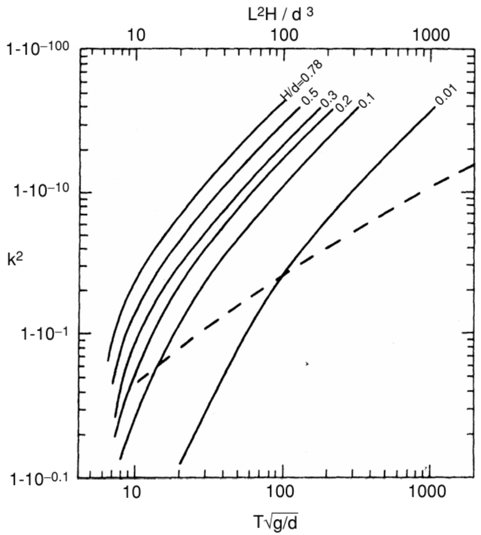

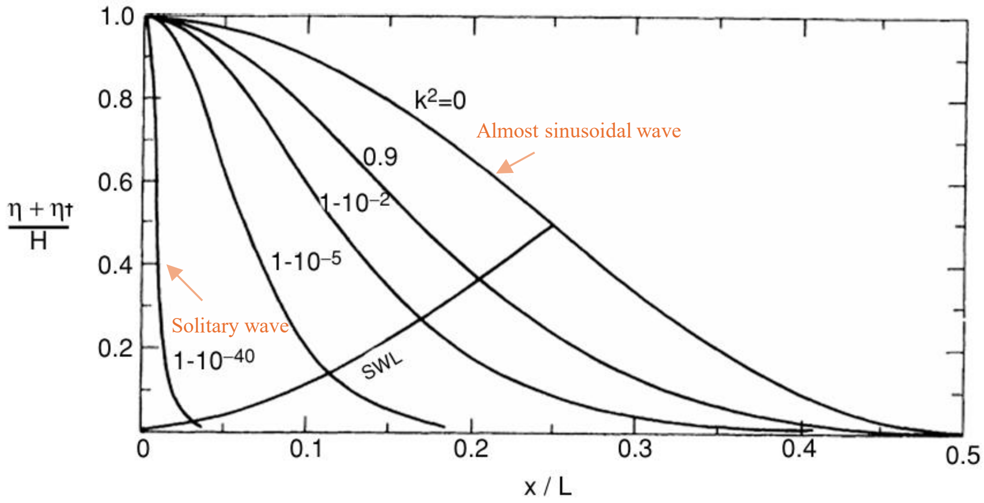

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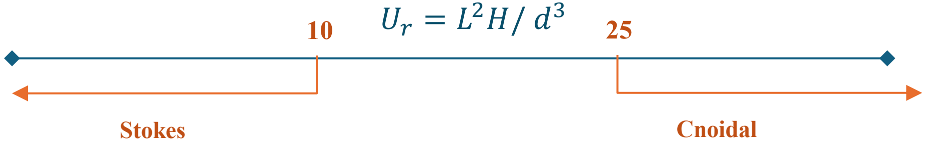
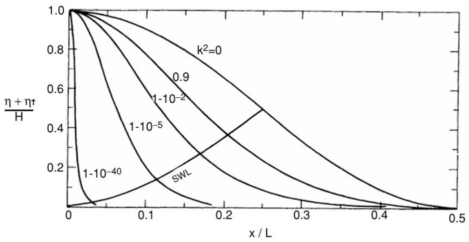


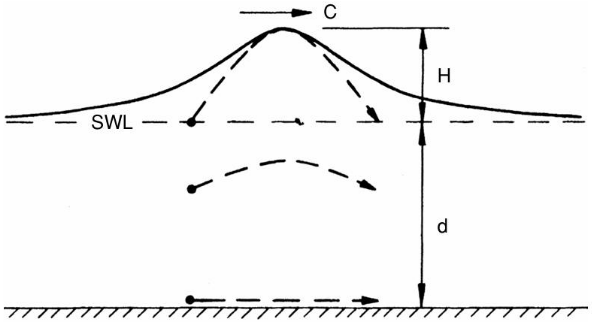
Solitary Waves

A solitary wave has a crest that is completely above the still water level, and no trough.

- The water particles move forward and then come to rest without returning to complete an orbit.
- Thus, it is a translatory rather than an oscillatory wave.
- It has an infinite wave length and period.

The surface profile is given by the limit $k^2 \rightarrow 1$





As a long period oscillatory wave propagates in very shallow water of decreasing depth, the surface profile approaches the solitary wave form. But the wave will break before a true solitary form is reached.

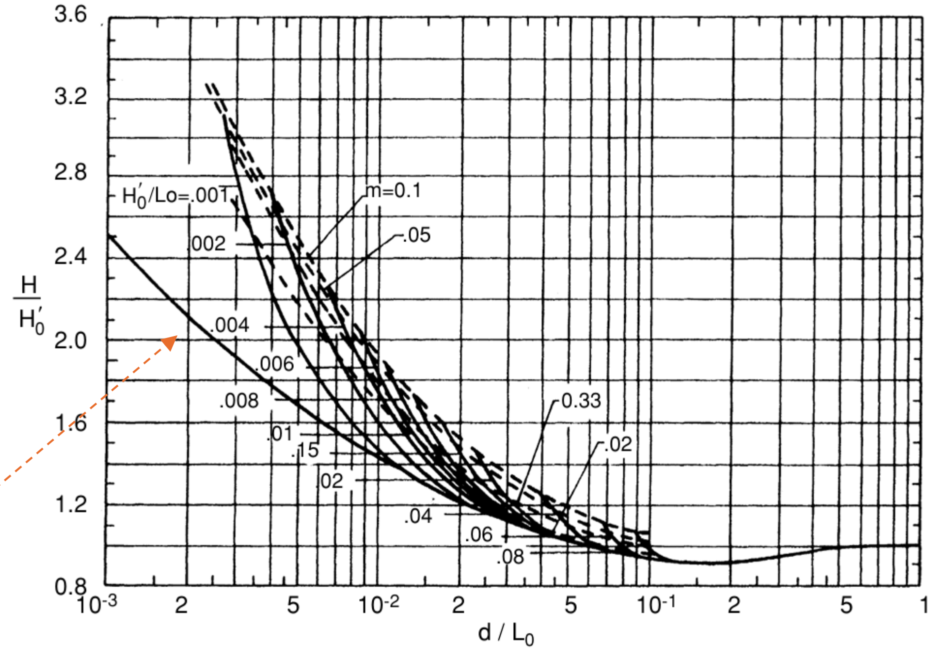
The cnoidal wave theory would still be most appropriate for these very long oscillatory waves in shallow water. However, owing to the complexity of cnoidal theory, solitary wave theory has been used by some investigators to calculate wave characteristics in very shallow relative water depths.

$$\eta = H\ \mathrm{sech}^2\left[\sqrt{\frac{3H}{4d^3}}(x - Ct)\right] \qquad E = \frac{8}{3\sqrt{3}}\rho g\ (Hd)^{3/2}$$

Shoaling effect in finite-amplitude theory

For steeper waves the small-amplitude wave theory is generally less valid. Finite-amplitude wave theories should be used to calculate changes in wave height as a wave propagates from one water depth to another.
A number of efforts have been made to employ finite-amplitude theories for wave shoaling analysis.

Walker and Headland (1982) evaluated various finite-amplitude theory approaches to shoaling analysis along with the available experimental data on wave shoaling and breaking to develop this figure →



Small-amplitude Shoaling line

The solid lines give the shoaling curves for increasing deep water wave steepnesses. The dashed lines indicate the breaker point as the waves shoal, for various beach slopes (i.e., m=0.05 is a slope of 1:20).