Lesson 5 - 05/03/2026

Fundamental theorem of calculus (Lebesgue's version)

Th. Let $g \in L^1(a,b)$, define $G(x) = \int_{[a,x]} g$

then G is a.e. differentiable and $G' = g$ a.e.

Lemma 1. Let $f : [a,b] \to \mathbb{R}$ increasing

then f' (f' exists a.e. from Lebesgue's diff. theorem)

$f' \in L^1(a,b)$

and $\int_{[a,b]} f \leq f(b) - f(a)$.

Lemma 2. Let $f \in L^1(a,b)$ and

suppose $\forall x \in [a,b]$, $\int_{[a,x]} f = 0$

then $f = 0$

proof of the theorem

It is not restrictive to consider $f \geq 0$

(if not $f = f^+ - f^-$ $f^+, f^- \geq 0$ and we apply the th. to f^+ and f^-)

step 1 let $0 \leq f \leq M$

consider $g_n(x) = n\left(F(x+\frac{1}{n}) - F(x)\right)$

where $F(x) = \int_{[a,x]} f$

we know that $\lim_n g_n(x) = F'(x)$ a.e.

and $|g_n(x)| = \left|n \int_x^{x+\frac{1}{n}} f(t)\,dt\right| \leq n \int_x^{x+\frac{1}{n}} |f(t)|\,dt$

$\leq n \cdot \int_x^{x+\frac{1}{n}} M\,dt = M$

$|g_n(x)| \leq M$

dominated conv.

$$\boxed{\lim_n \int_{[a,x]} g_n(t)\,dt = \int_{[a,x]} F'(t)\,dt} \quad (1)$$

$$\int_{[a,x]} g_n(t)\,dt = \int_a^x nF(t+\tfrac{1}{n}) - nF(t)\,dt$$

$$= n\left(\int_{a+\frac{1}{n}}^{x+\frac{1}{n}} F(t)\,dt - \int_a^x F(t)\,dt\right)$$

$$= \frac{\int_x^{x+\frac{1}{n}} F(t)\,dt}{\frac{1}{n}} + \frac{\int_a^{a+\frac{1}{n}} F(t)\,dt}{\frac{1}{n}}$$

$\downarrow n \to +\infty$ $F(x)$ $\quad\quad$ $\downarrow n \to +\infty$ $F(a)$

F is continuous (integral mean th. for cont. funct)

$$\boxed{\lim_n \int_{[a,x]} g_n(t)\,dt = F(x) - \underbrace{F(a)}_{0} = F(x)} \quad (2)$$

(1) + (2) $\Rightarrow$ $F(x) = \int_{[a,x]} F'(t)\,dt$

$\quad\quad F(x) = \int_{[a,x]} f(t)$

$$\int_{[a,x]} \left(f(t) - F'(t)\right)dt = 0 \quad \forall x$$

Lemma 2 $\Rightarrow$ $f = F'$ a.e.

Step 2 $\quad f$ not bounded

define $f_n(x) = \begin{cases} f(x) & \text{if } 0 \le f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$

$$f(x) = f_n(x) + (\underbrace{f(x) - f_n(x)}_{\geq 0})$$

$\lim_n f_n(x) = f(x)$ and $|f_n| \le |f|$

so $\lim_n \underbrace{\int_{[a,x]} f_n(t)\,dt}_{F_n(x)} = \int_{[a,x]} f = F(x)$

where $F'(x) = F_n'(x) + \left(\int_{[a,x]} f - f_n \right)'$

↑ increasing funct

$F'(x) \geq F_n'(x) = f_n(x)$ $\quad (\ )' \geq 0$

previous step

so $F'(x) \geq f_n(x)$ a.e. $\forall n$

conclusion $F'(x) \geq f(x)$ a.e. $\quad$ Lemma 1

so $F(x) = \int_{[a,x]} f(t) \leq \int_{[a,x]} F'(t) \leq F(x) - \underset{0}{\underset{\shortparallel}{F(a)}}$

$$F(x) \leq \int_{[a,x]} F'(t)\,dt \leq F(x)$$

so that $\quad \underset{\int_{[a,x]} f(t)}{\underset{\shortparallel}{F(x)}} = \int_{[a,x]} F'(t)\,dt$

So that, also in this case $\int_{[a,x]} f(t) - F'(t)\,dt = 0 \quad \forall x$

$\Downarrow$ ← Lemma 2

$f = F'$ a.e.

QED

Summary

$f \in L^+(a,b)$ and $F(x) = \int_{[a,x]} f$

then F is a.e. differentiable and $F' = f$ a.e.

Corollary

let $f \in L^+(a,b)$

then $\lim_{h \to 0} \dfrac{\int_{x-h/2}^{x+h/2} f(t)\,dt}{h} \left(= \lim_h \dfrac{F(x+\frac{h}{2}) - F(x-\frac{h}{2})}{h} \right) = f(x)$ for a.e. x

$\shortparallel$

$\lim_h \dfrac{F(x+h) - F(x)}{h}$

if for $x \in [a,b]$ this is valid

we say that x is a Lebesgue's point

we will prove this theorem for $f \in L^1(\Omega)$ — Ω open set in $\mathbb{R}^n$

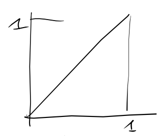
for a.e. $x \in \Omega$

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy = f(x)$$

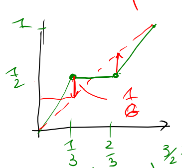
$$B(x,r) = \{ y \in \mathbb{R}^n : |x-y| < r \}$$

Ex. construct a function which is strictly increasing on $[0,1]$ and s.t. f is a.e. differentiable with $f' = 0$

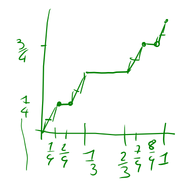
Ex. consider the Cantor function



$f_0(x) = x$



$$f_1(x) = \begin{cases} \frac{3}{2}x & 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} < x < \frac{2}{3} \\ \frac{3}{2}\left(x - \frac{2}{3}\right) + \frac{1}{2} & \frac{2}{3} \le x \le 1 \end{cases}$$



$$\|f_{n+1} - f_n\|_{L^\infty} = \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}$$

the set in which f_1 is constant $\left[\frac{1}{3}, \frac{2}{3}\right]$ — $\frac{1}{3}$

f_2 is const. $\left[\frac{1}{3}, \frac{2}{3}\right] \cup \left[\frac{1}{9}, \frac{2}{9}\right] \cup \left[\frac{7}{9}, \frac{8}{9}\right]$

$\frac{1}{3} + 2 \cdot \left(\frac{1}{9}\right)$

f_3 is con. $\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) +$

the limit is const. in a set with measure

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4\,\frac{1}{27} + \cdots 2^n \frac{1}{3^{n+1}} + \cdots$$

$$\frac{1}{3}\left(1 + \frac{2}{3} + \cdots \left(\frac{2}{3}\right)^n + \cdots\right) = \frac{1}{3}\sum_n \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = \frac{1}{3} \cdot 3 = 1$$

Ex. $f \in L^1(a,b)$

I say that $f \in BV([a,b])$ if $\exists\, \tilde{f} \in BV([a,b])$ s.t. $f = \tilde{f}$ a.e.

$$f \in L^1(a,b) \wedge f \in BV \Rightarrow \boxed{\exists M > 0 \,/\, \int_a^{b-\varepsilon} |f(t+\varepsilon) - f(t)|\,dt \le M\varepsilon}$$