Lesson 6 - 6/03/2026

Absolutely Continuous functions (AC functions)

def Consider $f:[a,b]\to\mathbb{R}$,
f is called absolutely continuous funct if
$\forall\varepsilon>0,\ \exists\delta>0$: given $(]\alpha_k,\beta_k[)_k$ (finite or countable)
with $]\alpha_k,\beta_k[\subseteq]a,b[$, pairwise disjoint sets

$$\sum_k(\beta_k-\alpha_k)<\delta \Rightarrow \sum_k|f(\beta_k)-f(\alpha_k)|<\varepsilon$$

first properties

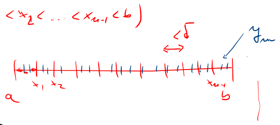
i) $f,g\in AC([a,b])$, $\lambda,\mu\in\mathbb{R}$ then $\lambda f+\mu g\in AC$

ii) $f\in AC([a,b])$ then $f\in\mathcal{C}([a,b])$

iii) $f\in AC([a,b])$ then $f\in BV([a,b])$

fix $\varepsilon=1$ in the def of AC there exists $\delta>0$ s.t. ....

take $\Delta=\{a=x_0,x_1,\dots x_{n-1},x_n=b\}$ $(a<x_1<x_2<\dots<x_{n-1}<b)$
subdivision of $[a,b]$
such that $x_j-x_{j-1}<\delta\quad\forall j$



take $\tilde\Delta$ another subdivision $\tilde\Delta=\{a=y_0<y_1<\dots<y_{m-1}<y_m=b\}$

$$V(f,\tilde\Delta)=\sum_{\ell=1}^{m}|f(y_\ell)-f(y_{\ell-1})|\le V(f,\Delta\cup\tilde\Delta)=$$

$$=\underbrace{\sum_{j=1}^{n}\Big(\underbrace{\sum_{\ell:\,x_{j-1}\le y_\ell\le x_j}|f(y_\ell)-f(y_{\ell-1})|}_{<1}\Big)}_{\le n}$$

$|f(x_{j-1})-f(y_\ell)|$ $|f(y_\ell)-f(x_j)|$

$V(f,\tilde\Delta)\le n$
↑ this is OK for all $\tilde\Delta$

$\Rightarrow V_a^b(f)\le n$.

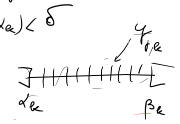
so $AC([a,b])\subseteq BV([a,b])\cap\mathcal{C}([a,b])$

iv) let $f\in AC([a,b])$ consider $x\longmapsto V_a^x(f)$
also $x\longmapsto V_a^x(f)$ is $AC([a,b])$

as a consequence every $AC([a,b])$ is the difference of two increasing $AC([a,b])$ function

let $f\in AC([a,b])$
I consider $\varepsilon>0$ and there $\delta>0$ st. ...
take $(]\alpha_k,\beta_k[)_k$ st. $\Big<$ pairwise disjoint; $\sum_k(\beta_k-\alpha_k)<\delta$

consider $V_{\alpha_k}^{\beta_k}(f)$



there exist $\alpha_k=y_{0,k}<y_{1,k}<\dots<y_{n_k-1,k}<y_{n_k,k}=\beta_k$

s.t. $V_{\alpha_k}^{\beta}(f)<\sum_{j=1}^{n_k}|f(y_{j,k})-f(y_{j-1,k})|+\frac{\varepsilon}{2^k}$

$$\sum_k V_{\alpha_k}^{\beta}(f)\le\sum_k\Big(\sum_j|f(y_{j,k})-f(y_{j-1,k})|+\frac{\varepsilon}{2^k}\Big)$$

$$\sum_k|V_a^{\beta_k}(f)-V_a^{\alpha_k}(f)|\le\underbrace{\sum_k\Big(\sum_j|f(y_{j,k})-f(y_{j-1,k})|\Big)}_{<\varepsilon}+\underbrace{\sum_k\frac{\varepsilon}{2^k}}_{<\varepsilon}<2\varepsilon$$

$(]y_{j-1,k},y_{j,k}[)_{j,k}$ $\Big<$ pair disjoint sets; $\sum\sum|y_{j,k}-y_{j-1,k}|<\delta$

$\forall \varepsilon > 0, \exists \delta > 0$: $(]\alpha_k, \beta_k[)_k$ s.t. $\sum_k \beta_k - \alpha_k < \delta$

$\Rightarrow \sum_k |V_a^{\beta_k}(f) - V_a^{\alpha_k}(f)| < 2\varepsilon \Rightarrow V_a^{+}(f) \in AC$

Remark Absolute continuity of the integral

Th. Suppose $(\Omega, \mathcal{F}, \lambda)$ is a measure space
take $f \in L^1_\lambda(\Omega)$
then $\forall \varepsilon > 0, \exists \delta > 0$: $\forall A \in \mathcal{F}$
if $\lambda(A) < \delta$ then $\int_A |f| \, d\lambda < \varepsilon$

proof. step 1 $|f| \le M$

to have $\int_A |f| d\lambda \le \lambda(A) \cdot M < \varepsilon$ it is suff. to take $\delta < \frac{\varepsilon}{M}$

step 2.
consider $f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n \\ n & \text{if } |f(x)| > n \end{cases}$

we have $\lim_n f_n(x) = f(x) \quad \forall x$
and $|f_n| \le |f|$
so that $\int_\Omega |f| \, d\lambda = \lim_n \int_\Omega |f_n|$

(or $\int_\Omega |f - f_n| \, d\lambda = 0$)

take $\varepsilon > 0$ consider $\bar{n}$ s.t. $\int_\Omega |f - f_{\bar{n}}| d\lambda < \varepsilon/2$

then $\int_A |f| \le \int_A |f_{\bar{n}}| + \int_\Omega |f_{\bar{n}} - f|$
$\le \varepsilon/2 + \varepsilon/2$ if $|A| \le \delta$

δ from f_n and $\varepsilon/2$

Consequence let $f \in L^1(a,b)$
consider $F(x) = \int_{[a,x)} f$
then F is $AC([a,b])$.

($\sum_k |F(\beta_k) - F(\alpha_k)| = \sum_k |\int_{[\alpha_k, \beta_k]} f| \le \sum_k \int_{[\alpha_k, \beta_k]} |f|$
$(A = \bigcup]\alpha_k, \beta_k[)$)

Characterization of $AC([a,b])$

Th
let $f \in AC([a,b])$
then $\forall x \in [a,b)$, $f(x) = f(a) + \int_{[a,x)} f'(t) dt$

(Rem $f \in AC \Rightarrow f \in BV$
$\Rightarrow f$ is a.e. diff.
and $f' \in L^1$)

Lemma let $f \in AC([a,b])$ and f increasing
suppose $f'(x) = 0$ for a.e. x
then f is a constant.

proof consider $H_1 = \{ x \in]a,b[\text{ s.t. } f'(x) = 0 \}$
$H_2 = \{ x \in]a,b[$ s.t. ~~$f'(x)$ does not exist~~ $\exists f'(x)$ and $f'(x) \ne 0 \}$

proof consider $H_1 = \{ x \in]a,b[\text{ s.t. } f'(x) = 0 \}$

$H_2 = \{ x \in]a,b[\text{ s.t. } f'(x) \text{ does not exist } \cup \exists f'(x) \text{ and } f'(x) \neq 0 \}$

Lebesgue measure $\longrightarrow \lambda(H_2) = 0$

take $\varepsilon > 0$ consider $f \in AC$, take the corresponding δ

consider $(]\alpha_k, \beta_k[)_k$ pairwise disjoint, $\sum \beta_k - \alpha_k < \delta$

and $H_2 \subseteq \bigcup_k]\alpha_k, \beta_k[$ (from regularity of Lebesgue measure)

then $\sum_k |f(\beta_k) - f(\alpha_k)| < \varepsilon$

$f(\beta_k) - f(\alpha_k) = \lambda(f(]\alpha_k, \beta_k[))$ f increasing

$\lambda(f(\bigcup]\alpha_k, \beta_k[)) < \varepsilon$

$f(H_2) \subseteq f(\bigcup]\alpha_k, \beta_k[)$

$\lambda(f(H_2)) < \varepsilon \quad \forall \varepsilon \Rightarrow \underline{\lambda(f(H_2)) = 0}$

take now $x \in H_1 = \{ x \in]a,b[: f'(x) = 0 \}$

in particular, taking $\varepsilon > 0$

$\exists \rho > 0 : \forall y \in]x-\rho, x+\rho[\setminus \{x\}$

$$0 \leq \frac{f(y) - f(x)}{y - x} < \frac{\varepsilon}{b-a}$$

in particular $\exists y > x$ s.t. $\frac{f(y) - f(x)}{y-x} < \frac{\varepsilon}{b-a}$

so $y \cdot \frac{\varepsilon}{b-a} - f(y) > x \cdot \frac{\varepsilon}{b-a} - f(x)$

for $y > x$

$x \in H_1 \Rightarrow x \in I_\varepsilon$ = invisible points from right for $x \mapsto \frac{x\varepsilon}{b-a} - f(x)$

$I_\varepsilon = \bigcup_k]\alpha_k, \beta_k[$

and $\beta_k \frac{\varepsilon}{b-a} - f(\beta_k) \geq \alpha_k \frac{\varepsilon}{b-a} - f(\alpha_k)$

so that $f(\beta_k) - f(\alpha_k) \leq (\beta_k - \alpha_k) \cdot \frac{\varepsilon}{b-a}$

so that $\sum_k f(\beta_k) - f(\alpha_k) \leq \left(\underbrace{\sum_k (\beta_k - \alpha_k)}_{b-a} \right) \frac{\varepsilon}{b-a} = \varepsilon$

in conclusion $H_1 \subseteq I_\varepsilon = \bigcup]\alpha_k, \beta_k[$

but $\lambda(f(I_\varepsilon)) = \sum_k (f(\beta_k) - f(\alpha_k)) \leq \varepsilon$

so $\lambda(f(H_1)) < \varepsilon \quad \forall \varepsilon$

$\boxed{\lambda(f(H_1)) = 0}$ remember $\boxed{\lambda(f(H_2)) = 0}$

$\lambda(f([a,b])) = 0$ f continuous $\Rightarrow$ $\underline{f \text{ constant}}$

Th. let $f \in AC([a,b])$

then $\forall x \in [a,b]$, $f(x) = f(a) + \int_a^x f'(t)\,dt$

proof. It is not restrictive to consider f increasing

define $g(x) = f(x) - \int_a^x f'(t)\,dt$

take $x_1, x_2 \in [a,b]$ s.t. $x_1 < x_2$

consider $g(x_2) - g(x_1) = f(x_2) - f(x_1) - \left(\int_a^{x_2} f'(t) - \int_a^{x_1} f'(t) \right)$

$= f(x_2) - f(x_1) - \int_{x_1}^{x_2} f'(t)\,dt$ ≥ 0

but f is increasing and then $\underbrace{\int_{x_1}^{x_2} f'(t)\,dt \leq f(x_2) - f(x_1)}_{\text{Fatou}}$

g is AC and g is increasing

and $g'(x) = \left(f(x) - \int_a^x f'(t)\,dt \right)' = f'(x) - f'(x) = 0$

the lemma gives $g =$ constant.

so that $f(x) - \int_a^x f'(t)\,dt = \text{constant} \overset{!}{=} f(a) - 0 = f(a)$

QED

proposition ~~Corollary~~ let $F, G \in AC([a,b])$

then $F \cdot G$ is $AC([a,b])$

and $(FG)'(x) = F'(x)G(x) + F(x)G'(x)$ for a.e. x

proof F and G are a.e. diff.

$(FG)' = F'G + FG'$ in the points in which F and G are diff.

It remains to prove that FG is in AC

$\sum_k |F(\beta_k)G(\beta_k) - F(\alpha_k)G(\alpha_k)|$

$\leq \sum_k |F(\beta_k)G(\beta_k) - F(\beta_k)G(\alpha_k)| + |F(\beta_k)G(\alpha_k) - F(\alpha_k)G(\alpha_k)|$

$\leq \sup|F| \cdot \sum_k |G(\beta_k) - G(\alpha_k)| + \sup|G| \sum_k |F(\beta_k) - F(\alpha_k)|$

Corollary let $f, g \in L^1(a,b)$

define $F(x) = \int_a^x f(t)\,dt$, $G(x) = \int_a^x g(t)\,dt$

then $\int_{[a,b]} fG = F(b)G(b) - \int_{[a,b]} Fg$

Remark consider $F \in AC([a,b])$

consider $\varphi \in C_c^1(]a,b[)$ — $\varphi \in C^1$, supp φ is a compact contained in $]a,b[$

then $\int_a^b F\varphi' = -\int_a^b F'(t)\varphi$ and $F' \in L^1$

Remark

consider $F \in AC([a,b])$

consider $\varphi \in \mathcal{C}^1_0(]a,b[)$ — $\varphi \in \mathcal{C}^1$, supp φ is a compact contained in $]a,b[$

then $\displaystyle\int_a^b F\varphi' = -\int_a^b F'(t)\varphi$ and $F' \in L^1$

as a consequence $AC([a,b]) \subseteq W^{1,1}(]a,b[)$

where $W^{1,1}(]a,b[) = \Big\{ h \in L^1(a,b)$ s.t. $\exists g \in L^1(a,b)$ s.t. $\forall \varphi \in \mathcal{C}^1_0(]a,b[)$ $\displaystyle\int_a^b h\varphi' = -\int_a^b g\varphi \Big\}$

At the end $\boxed{AC([a,b]) = W^{1,1}(]a,b[)}$

Remark

let $g : [a,b] \to \mathbb{R}$

we say that g has the property (N) if $\forall A \in]a,b[$ s.t. $\lambda(A) = 0$ then

$$\lambda(g(A)) = 0$$

Th. let g continuous and $g \in BV$.

g has the property $(N) \iff g \in AC([a,b])$.

---

it is possible to prove that, for $f \in L^1(a,b)$

$f \in BV \iff \exists M_f$ s.t. $\forall \varepsilon > 0$ $\displaystyle\int_a^{b-\varepsilon} |f(t+\varepsilon) - f(t)|\,dt \le M\varepsilon$

weaker than

suppose $a(t) \geq t_0 > 0$

suppose $\exists M$ s.t. $\forall \varepsilon > 0$ $\displaystyle\int_a^{b-\varepsilon} |a(t+\varepsilon) - a(t)|\,dt \le C\varepsilon\,|\log|\varepsilon||$

then the Cauchy problem for $\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = f \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1 \end{cases}$

is well posed in $\mathcal{C}^\infty$.