# Standard Model - Problem sheet

Enrico Morgante*

April 16, 2026

## 1st Week: Lagrangians and Symmetries

### Exercise 1.1: Natural Units

Given the constants used in the natural units convention,

$$c \approx 3 \times 10^8 \,\mathrm{m/s} = 1\,, \quad \hbar \approx 6.626 \times 10^{-34}\,\mathrm{J\,s}\,, \quad \hbar c \approx 197.3\,\mathrm{MeV\,fm} = 1 \tag{1.1}$$

solve the following exercises.

1. The width of a particle is defined as the inverse of its litefime. The mean lifetime for the $B^+$ meson is $\tau \approx 1.64 \times 10^{-12}\,\mathrm{s}$. What is its width in eV?

2. Find the average distance traveled in the lab frame by a particle with $\gamma = 100$ and a decay width of $\Gamma = 2.3\,\mathrm{eV}$;

3. Quantum gravity effects cannot be neglected at very short distances. This happens when the energy scale is of the order of the Planck mass:

$$M_P = \sqrt{\frac{\hbar c}{G_N}} \tag{1.2}$$

   where $G_N$ is the Newtonian gravitational constant. Express $M_P$ in GeV, and the Planck length $L_P = M_P^{-1}$ in centimeters.

4. In oscillation experiments for neutrinos, it is important to know the oscillation length, $L_{osc} = 4\pi E/\Delta m^2$, where $\Delta m^2$ is the mass-squared difference between the two neutrino states. For an experiment conducted with neutrinos of $E = 1.3\,\mathrm{GeV}$, find the value of $\Delta m^2$ in units of $\mathrm{eV}^2$ that corresponds to $L_{osc} = 140$ meters.

### Exercise 1.2: Weyl & Dirac spinors

Use the chirality projectors to rewrite the following Langragian in function of 4-components Weyl spinors, and then in function of 2-components Weil spinors in the Weil basis:

$$\mathscr{L}_F = i\bar{\psi}\partial\!\!\!/\psi - m_D\bar{\psi}\psi - e\bar{\psi}A\!\!\!/\psi \tag{1.3}$$

*I warmly thank Raffaele Tiede for the help in preparation of this document.

## Exercise 1.3: Gamma matrices

Prove, without relying on any explicit representation, the following identities

1. $\gamma^5 \equiv -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3$, where $\epsilon^{0123} = +1$ and $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$
2. $(\gamma^5)^2 = \mathbb{1}$
3. $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$
4. $\gamma_\mu \not{p}\not{q}\not{p}\gamma^\mu = -2\not{p}\not{q}\not{p}$
5. $\{\gamma^5, \gamma^\mu\} = 0$
6. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

## Exercise 1.4: Fierz identities

Consider the basis of $4 \times 4$ matrices

$$\Gamma^A = \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, i\gamma^\mu\gamma^5, i\gamma^5\} \tag{1.4}$$

and its hermitian conjugate

$$\Gamma_A = \{\mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, -i\gamma_\mu\gamma^5, -i\gamma^5\}\,. \tag{1.5}$$

Any $4 \times 4$ matrix can be expressed as

$$M = \sum_A c_A \Gamma^A\,. \tag{1.6}$$

a) Find the value of the coefficients $c_A$ for a generic matrix $M$.

b) Prove the completeness relation

$$\sum_A \frac{1}{4}(\Gamma^A)_{ij}(\Gamma_A)_{kl} = \delta_{il}\delta_{kj} \tag{1.7}$$

c) Prove the following Fierz identity:

$$(\bar\psi_1\gamma^\mu P_L\psi_2)(\bar\psi_3\gamma_\mu P_L\psi_4) = -(\bar\psi_1\gamma^\mu P_L\psi_4)(\bar\psi_3\gamma_\mu P_L\psi_2)\,. \tag{1.8}$$

Similar identities hold for scalar, pseudoscalar, vector and axial-vector bilinears, and can be obtained with the same techniques used in this exercise.

# $2^{\text{nd}}$ Week: Symmetries

## Exercise 2.1: Discrete symmetries

Derive the transformation properties of the spinor bilinears $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\partial\!\!\!/\psi$ under the discrete transformations $P, C, CP$. *Suggestion:* in some cases, you will need to consider not just the bilinear $\mathcal{O}$, but its contribution to the action $\mathcal{S} = \int d^4x\ \mathcal{O}$.

## Exercise 2.2: Accidental symmetries

Let us consider a model with two scalar fields $\phi_1, \phi_2$, charged under a gauged $U(1)_{\text{gauge}}$ with charges $q_1 = 4$, $q_2 = 1$. (NB: the label "gauge" is only a label. From the point of view of group theory, this is just $U(1)$.)

1. Write down the most generic renormalizable Lagrangian for the model.
2. The model has a trivial global symmetry and a non-trivial one. What are they?
3. Now include in the Lagrangian higher order terms up to mass dimension 6. What is the fate of global symmetries?

This problem is a simplified version of what happens in the SM as well. The baryon and the lepton numbers are *accidental symmetries* of the SM: they are not imposed from the start, but they are consequences of the renormalizability of the Lagrangian and of the charge (representation) assignment under the gauge group. If one includes non-renormalizable terms, interpreted as low-energy limit of some heavy new physics, the baryon and lepton numbers may be violated.

# $3^{\text{rd}}$ Week: Abelian gauge symmetry and QED

## Exercise 3.1: Chiral Symmetry

Consider a Dirac fermion $\psi$ and the following transformations:

$$\psi \to e^{i\theta}\psi \quad \text{and} \quad \psi \to e^{i\theta\gamma_5}\psi \tag{3.1}$$

1. Find the transformation laws of $\bar{\psi}$. Hint: for the $\gamma_5$ transformation, use a Taylor expansion of the exponential.
2. Show that the Lagrangian of a massless, free Dirac fermion is invariant under Eq. (3.1)
3. Find the transformation for the left and right Weyl spinors under Eq. (3.1), demonstrating that $e^{i\theta\gamma_5}\psi$ is an axial transformation.
4. Is the Dirac mass term $m\bar{\psi}\psi$ invariant under (3.1)?.

Now consider a system of $N$ fermions $\psi_i$ that carry the same charge under $U(1)$, interacting with a neutral scalar $\phi$:

$$\mathscr{L} = \bar{\psi}_i[i\partial\!\!\!/\delta_{ij} - m_{ij} - \lambda_{ij}\phi]\psi \tag{3.2}$$

5. For generic $m_{ij}, \lambda_{ij}$ there is a U(1) symmetry. What is the symmetry group if $\lambda_{ij} \propto m_{ij}$?
6. Find a transformation for $\psi_{L,R}$, $\phi$ that imposes $m_{ij} = 0$ but allows $\lambda_{ij} \neq 0$. Explain why the symmetry must be chiral.

## Exercise 3.2: Covariant Derivative & Gauge Fields

1. Show explicitly that the covariant derivatives $D_\mu\phi$ and $D\!\!\!\!/\,\psi$ transform as the field themselves under a gauge transformation.
2. Show that $F_{\mu\nu}$ is gauge invariant.
3. Show that $[D_\mu, D_\nu]\psi = igq\, F_{\mu\nu}\psi$. This relation can be used as an alternative definition of $F_{\mu\nu}$.
4. Show that $F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ is a total derivative, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor with $\epsilon^{0123} = +1$.

## Exercise 3.3: Compton scattering

1. Draw the Feynman diagrams for the scattering process $e^-\gamma \to e^-\gamma$. At tree level there are two diagrams. In one, the exchanged momentum is $p_1^\mu + p_2^\mu$, in the other $p_2^\mu - p_4^\mu$. From the definition of the Mandestalm variables $(p_1+p_2)^2 = (p_3+p_4)^2 \equiv s$, $(p_1-p_3)^2 = (p_2-p_4)^2 \equiv t$ and $(p_1 - p_4)^2 = (p_2 - p_3)^2 \equiv u$, the two diagrams are said to be in the $s-$channel and in the $t-$channel, respectively.
2. Compute the matrix element.

3. Square the matrix element, sum over the final spin/polarization states, and average over the initial one. You can use the replacement

$$\sum_{\text{pols. } i} \epsilon_\mu^{i\,*} \epsilon_\nu^i \to -g_{\mu\nu} \,. \tag{3.3}$$

After some work you should find

$$\frac{1}{4} \sum_{\text{spins/pols.}} |\mathcal{M}|^2 = 2e^4 \left[ \frac{p_{24}}{p_{12}} + \frac{p_{12}}{p_{24}} + 2m^2 \left( \frac{1}{p_{12}} - \frac{1}{p_{24}} \right) \right] , \tag{3.4}$$

where $p_{ij} = (p_i)_\mu (p_j)^\mu$. Hint: you will find traces of six gamma matrices, which can be simplified using $\gamma^\nu \not{p} \gamma_\nu = -2\not{p}$.

We want to evaluate the cross section in the lab frame, in which the electron is initially at rest.

4. Writing the momenta as

$$\begin{aligned} p_1 &= (\omega, 0, 0, \omega) & p_2 &= (m_e, 0, 0, 0) \\ p_4 &= (\omega', \omega' \sin\theta, 0, \omega' \cos\theta) & p_3 &= p_1 + p_2 - p_4 = (E', \vec{p}') \end{aligned} \tag{3.5}$$

show that $\omega' = \frac{\omega}{1 + \frac{\omega}{m_e}(1 - \cos\theta)}$, and thus, for the wavelength,

$$\Delta\lambda = \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m_e}(1 - \cos\theta) \tag{3.6}$$

which is Compton's formula for the wavelength shift. No QED is involved here: just relativistic kinematics.

5. Evaluate $d\Pi_{\text{LIPS}}$ in the lab frame, and use it to compute the cross section. You should obtain the *Klein-Nishina formula*

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2} \left( \frac{\omega'}{\omega} \right)^2 \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right] . \tag{3.7}$$

6. Finally, go to the non-relativistic limit $m_e \gg \omega$. Obtain the Thompson scattering cross section for the classical scattering of light from a free electron:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2} [1 + \cos^2\theta] \,. \tag{3.8}$$

# $4^{\text{th}}$ Week: QED

## Exercise 4.1: Anomalous magnetic moment

This problem concern the derivation of the (anomalous) magnetic moment of the electron. This is one of the real milestones of the development of QFT, and was decisive to convince people of the power of QED. We will work at the border between QFT and relativistic quantum mechanics, without discussing in details how to translate our result into the Schroedinger picture of QM. You can find more details on this point in the Lecture Notes by Ben Simons (Cambridge): https://www.tcm.phy.cam.ac.uk/ bds10/aqp.html

1. Start from the Dirac equation of relativistic quantum mechanics: $(i\not\partial - e\not A - m)\psi = 0$. Multiply it times $(i\not\partial - e\not A + m)$ on the left and obtain the equation

$$\left((i\partial_\mu - eA_\mu)^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\right)\psi = 0 \tag{4.1}$$

2. Compute explicitly the term $\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$ in terms of the $\vec{E}$ and $\vec{B}$ fields and show that this term leads to a magnetic dipole interaction $\vec{S}\cdot\vec{B}$.[1]

3. Derive the Hamiltonian by going to Fourier space, replacing $i\partial_t\psi = H\psi$ and $i\partial_i = p_i\psi$, and then realizing that, in the non-relativistic limit,

$$H\psi \approx (m + H_{\text{int}})\psi \qquad \text{and} \qquad (H - eA_0)^2\psi \approx [m^2 + 2m(H_{\text{int}} - eA_0)]\psi \tag{4.2}$$

Now you should be able to read off from your Hamiltonian the magnetic dipole interaction

$$g\frac{e}{2m}\vec{B}\cdot\vec{\sigma}\,\psi \tag{4.3}$$

with $g = 2$.

From the QM point of view, $g$ is a parameter that could take any value. Explaining the experimental observation $g \approx 2$ was one of the great successes of the Dirac equation. $g = 2$ comes from the prefactor $e/2$ in front of $F_{\mu\nu}\sigma^{\mu\nu}$ in Eq. (4.1): were it $g'$, we would have had $g = g'$. Thus, a useful way of finding corrections to $g = 2$, is to look for loop diagrams that have the same effect of an additional $F_{\mu\nu}\sigma^{\mu\nu}$ term. The relevant process to look at is the scattering of an electron off an off-shell, classical photon field $A_\mu$:[2]

$$e^-(q_1)A_\mu(p) \to e^-(q_2)\,. \tag{4.4}$$

4. Start from the tree level matrix element, with the usual spinors $u, \bar{u}$ for the external electrons and writing $A^\mu_{\text{cl}}$ for the vector field. Using the Gordon identity, show that it can be written in the form

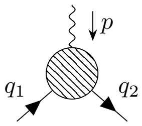
$$\mathcal{M} = A_{\mu\,\text{cl}}\mathcal{M}^\mu = \text{stuff } - \frac{e}{2m}i\bar{u}(q_2)\sigma^{\mu\nu}p_\nu u(q_1)A_{\mu\,\text{cl}} \tag{4.5}$$

The second term is the one responsible for $g = 2$ above.

[1] You will find an imaginary term proportional to the electric field: it does not represent an electric dipole, but it is just there because of Lorentz covariance. In a fixed reference frame you can simply set it to zero.

[2] Wait, what? A classical, off-shell field? Aren't all fields quantized? Aren't all external legs on-shell? You will understand this better (or simply get used to that) if you continue studying QFT!

5. Go to the one loop level. Draw the diagram responsible for the transition.
6. This 1-loop matrix element can be written as




$$= i\bar{u}(q_2)\Gamma^\mu u_(q_1)A_\mu(p) \tag{4.6}$$

   Without computing the loop diagram, find the most generic form of $\Gamma^\mu$ as a sum of four terms, each of them is a vector (including $\gamma^\mu$) with index $\mu$ times a coefficient that depends only on the scalar products of the same four vectors. Using momentum conservation at the vertex, the fact that electrons are on-shell and the Dirac equation, show that the coefficient depend on $p^2$ only.

7. Now use the Ward and the Gordon identity to prove that $\mathcal{M}$ can be written in the form

$$i\mathcal{M} = i\bar{u}(q_2)\left[F_1(p^2)\gamma^\mu + F_2(p^2)F_{\mu\nu}\sigma^{\mu\nu}\right]u_(q_1)A_\mu(p) \tag{4.7}$$

The factor $F_1$ is responsible for the renormalization of the electric charge. The factor $F_2$ instead gives the *anomalous* magnetic moment of the electron, *ie* the deviation of $g$ from 2:

$$g = 2 + 2F_2(0)\,. \tag{4.8}$$

$F_2(0)$ can be computed performing the loop integral. It turns out to be finite, and the result is $F_2(0) = \alpha/(2\pi)$, ie

$$g = 2 + \frac{\alpha}{\pi}\,. \tag{4.9}$$

# $5^{\text{th}}$ Week: QCD

## Exercise 5.1: Non-Abelian Gauge Fields

Let us consider a theory invariant under a local $SU(N)$ symmetry:

1. Starting from the covariant derivative, show that, for any representation $t^a_R$ of the gauge group, $\frac{i}{g}[D_\mu, D_\nu] = G^a_{\mu\nu} t^a_R$.
2. Find how the gauge fields must transform from the transformation law of the covariant derivative; what is the infinitesimal transformation?
3. Derive the transformation law of the field strength $G^a_{\mu\nu}$
4. Derive the Noether currents of the theory including fermions.

## Exercise 5.2: Inclusive $e^+e^- \to \text{hadrons}$ cross section

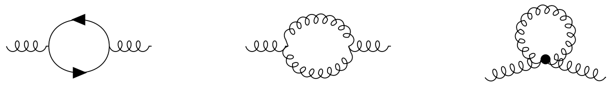
Starting from the Feynman rules, compute at tree level

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{5.1}$$

where $\sigma(e^+e^- \to \text{hadrons}) \approx \sigma(e^+e^- \to q\bar{q})$ at tree level. You can use the result for $\sigma(e^+e^- \to \mu^+\mu^-)$ derived in the lectures.

## Exercise 5.3: Loop corrections and group representation matrices

Write down the corrections to the gluon propagator corresponding to the three Feynman diagrams



Do not compute the integrals, but take care of the necessary contractions of SU(3) generators.

# $6^{\text{th}}$ Week: Spontaneous symmetry breaking

## Exercise 6.1: Goldstone boson scattering

Consider the Lagrangian of a complex scalar field:

$$\mathcal{L} = |\partial_\mu \phi|^2 + \mu^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 \tag{6.1}$$

We will consider two different parametrization of the fields after spontaneous symmetry breaking, and show that, although the Lagrangians are different, they describe the same physics.

1. Use the polar decomposition

$$\phi(x) = \left( v + \frac{\sigma(x)}{\sqrt{2}} \right) e^{i\pi(x)/F_\pi} \tag{6.2}$$

   with real $\sigma, \pi$ and rewrite the Lagrangian. Show that, if $v$ is chosen to minimize the potential, the linear term in $\sigma$ cancels. Find the correct value of $F_\pi$ and the masses of $\pi$ and $\sigma$.

2. Now use the linear decomposition

$$\phi(x) = v + \frac{\chi_1 + i\chi_2}{\sqrt{2}} \, . \tag{6.3}$$

   Check that the normalization of the real fields $\chi_i$ is correct. Rewrite the Lagrangian and show that the masses are the same as above. This is the first check that the two theories are equivalent: the mass spectrum is the same.

3. Write the relation between $\sigma, \pi$ and $\chi_1, \chi_2$ as a series. With a simple drawing on cartesian axes, identify the fields to show that they represent the same fluctuations at linear order.

4. Compute the Feynman rule of the $\pi\pi\sigma$ vertex in the polar theory. You can use the notes on Feynman diagrams that you find on Moodle.

5. Compute the Feynman rules of the $\chi_2^4$ and $\chi_2^2\chi_1$ vertices in the linear theory.

6. Show that the matrix element for the $\pi\pi \to \pi\pi$ and $\chi_2\chi_2 \to \chi_2\chi_2$ scatterings are the same, once the external particles are taken on-shell. It is convenient to use Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2 \tag{6.4}$$

   and the relation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 = 4m_\pi^2 = 0 \, . \tag{6.5}$$

This is a simple case of a more general theorem, that ensures the equivalence of the scattering matrix under a broad class of field redefinitions, as long as the field operators create the same particles from the vacuum. For more details you can look at Weinberg vol. 1, ch. 7.

## Exercise 6.2: The linear sigma model

In this problem we will construct a phenomenological Lagrangian for the interactions of pions with nucleons in QCD. This model predates QCD itself, and was developed before the introduction of quarks. Nowadays, we understand it as an effective theory of strong interactions at low energies.

We consider a theory with a spontaneously broken global symmetry:

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B \to \mathrm{SU}(2)_V \times \mathrm{U}(1)_B\,. \tag{6.6}$$

There are two Weyl fermions for the left and right-handed nucleon doublets

$$N_L(2,1)_1 = \begin{pmatrix} p_L \\ n_L \end{pmatrix}\,, \qquad N_R(1,2)_1 = \begin{pmatrix} p_R \\ n_R \end{pmatrix}\,. \tag{6.7}$$

We introduce a scalar field $\Sigma$, that will describe into the Lagrangian 3 Goldstone bosons and a fourth scalar responsible for SSB.

1. Show that, to allow a renormalizable interaction term of $\Sigma$ with $N_L$ and $N_R$, $\Sigma$ must transform as a 2 under $\mathrm{SU}(2)_L$ and as a $\bar{\mathbf{2}}$ under $\mathrm{SU}(2)_R$. Having two indices, you can think of $\Sigma$ as a matrix $\Sigma_{a\bar{b}}$, where $\bar{b}$ is a $\mathrm{SU}(2)_R$ index. Write down the transformation law of $\Sigma$ in matrix notation and an interaction term for $\Sigma N_L N_R$.

$\Sigma$ as a $2\times 2$ complex matrix has 8 real dof. We want to impose a reality condition that reduces the dof to 4. We first need to convince ourselves that the reality condition is not spoiled by transformations.

2. Show that $\tilde{\Sigma} \equiv \sigma^2 \Sigma^* \sigma^2$ transforms in the same representation of $\Sigma$. Use that $\sigma^2 U \sigma^2 = U^*$ for any $U \in \mathrm{SU}(2)$.

3. Expand $\Sigma$ in the complex basis $\{\mathbb{1}, \sigma^a\}$ and impose the *reality* condition $\Sigma = \tilde{\Sigma}$. Conclude that it is consistent to assume

   $$\Sigma(x) = s(x)\mathbb{1} + i\pi^a(x)\frac{\sigma^a}{2} \tag{6.8}$$

   for *real* $s, \pi^a$.

4. Write down a Lagrangian for the $\Sigma$ field, including mass and quartic terms, using traces to obtain invariant terms. Remember that the mass term in $\mathcal{L}$ must have a + sign to induce spontaneous symmetry breaking. You should have two quartic terms: using the decomposition in Eq. 6.8 show that they are equivalent, and you can keep one of them by rescaling the coupling $\lambda$.

5. Write the field as $\Sigma(x) \sim (F_\pi + \sigma(x))\mathbb{1} + i\pi^a(x)\sigma^a$ and write the Lagrangian (you should find the correct normalization). Rewrite the Lagrangian in terms of these fields. At the very end you should find, neglecting nucleons,[3]

   $$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi^a)^2 + \frac{1}{2}m^2\sigma^2 - \lambda F_\pi \sigma(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \tag{6.9}$$

   *Hint:* start by simplifying $\Sigma^\dagger\Sigma$. You should have found two quartic terms: expanding $\Sigma$ you can show easily that they are related to each other

[3] This is called the *linear sigma-model*: the symmetry acts linearly on $\Sigma$, and the order parameter is called $\sigma$.

6. Put nucleons back. Expand the Lagrangian and extract the mass of all the particles. You should find the *Goldberger-Treiman* relation:

$$m_N = g_{\pi NN} F_\pi \,, \tag{6.10}$$

where $g_{\pi NN}$ is the coupling constant of the $\bar{N}_L \Sigma N_R$ term.