Standard Model - Problem sheet

Enrico Morgante

May 6, 2026

The oral exam will consist of discussing the solutions to the proposed exercises, referring to the theory explained in class. The exercises marked with an asterisk (starting from week 8) are optional: solving them is not required to take the exam. However, I strongly recommend that you attempt at least some of them to help you better understand the theory discussed in class.

1st Week: Lagrangians and Symmetries

Exercise 1.1: Natural Units

Given the constants used in the natural units convention,

$$c \approx 3 \times 10^8\,\mathrm{m/s} = 1\,, \quad \hbar \approx 6.626 \times 10^{-34}\,\mathrm{J\,s}\,, \quad \hbar c \approx 197.3\,\mathrm{MeV\,fm} = 1 \tag{1.1}$$

solve the following exercises.

1. The width of a particle is defined as the inverse of its litefime. The mean lifetime for the B^+ meson is $\tau \approx 1.64 \times 10^{-12}$ s. What is its width in eV?

2. Find the average distance traveled in the lab frame by a particle with $\gamma = 100$ and a decay width of $\Gamma = 2.3\,\mathrm{eV}$;

3. Quantum gravity effects cannot be neglected at very short distances. This happens when the energy scale is of the order of the Planck mass:

$$M_P = \sqrt{\frac{\hbar c}{G_N}} \tag{1.2}$$

where G_N is the Newtonian gravitational constant. Express M_P in GeV, and the Planck length $L_P = M_P^{-1}$ in centimeters.

4. In oscillation experiments for neutrinos, it is important to know the oscillation length, $L_{osc} = 4\pi E/\Delta m^2$, where Δm^2 is the mass-squared difference between the two neutrino states. For an experiment conducted with neutrinos of $E = 1.3\,\mathrm{GeV}$, find the value of Δm^2 in units of eV^2 that corresponds to $L_{osc} = 140$ meters.

Exercise 1.2: Weyl & Dirac spinors

Use the chirality projectors to rewrite the following Langragian in function of 4-components Weyl spinors, and then in function of 2-components Weil spinors in the Weil basis:

$$\mathscr{L}_F = i\bar{\psi}\partial\!\!\!/\psi - m_D\bar{\psi}\psi - e\bar{\psi}A\!\!\!/\psi \tag{1.3}$$

Exercise 1.3: Gamma matrices

Prove, without relying on any explicit representation, the following identities

1. $\gamma^5 \equiv -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3$, where $\epsilon^{0123} = +1$ and $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$
2. $(\gamma^5)^2 = \mathbb{1}$
3. $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$
4. $\gamma_\mu \not{p}\not{q}\not{p} \gamma^\mu = -2\not{p}\not{q}\not{p}$
5. $\{\gamma^5, \gamma^\mu\} = 0$
6. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

Exercise 1.4: Fierz identities

Consider the basis of 4×4 matrices

$$\Gamma^A = \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, i\gamma^\mu\gamma^5, i\gamma^5\} \tag{1.4}$$

and its hermitian conjugate

$$\Gamma_A = \{\mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, -i\gamma_\mu\gamma^5, -i\gamma^5\}\,. \tag{1.5}$$

Any 4×4 matrix can be expressed as

$$M = \sum_A c_A \Gamma^A\,. \tag{1.6}$$

a) Find the value of the coefficients c_A for a generic matrix M.

b) Prove the completeness relation

$$\sum_A \frac{1}{4}(\Gamma^A)_{ij}(\Gamma_A)_{kl} = \delta_{il}\delta_{kj} \tag{1.7}$$

c) Prove the following Fierz identity:

$$(\bar\psi_1\gamma^\mu P_L\psi_2)(\bar\psi_3\gamma_\mu P_L\psi_4) = -(\bar\psi_1\gamma^\mu P_L\psi_4)(\bar\psi_3\gamma_\mu P_L\psi_2)\,. \tag{1.8}$$

Similar identities hold for scalar, pseudoscalar, vector and axial-vector bilinears, and can be obtained with the same techniques used in this exercise.

2^{nd} Week: Symmetries

Exercise 2.1: Discrete symmetries

Derive the transformation properties of the spinor bilinears $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\partial\!\!\!/\psi$ under the discrete transformations P, C, CP. *Suggestion:* in some cases, you will need to consider not just the bilinear $\mathcal{O}$, but its contribution to the action $\mathcal{S} = \int d^4x\ \mathcal{O}$.

Exercise 2.2: Accidental symmetries

Let us consider a model with two scalar fields ϕ_1, ϕ_2, charged under a gauged $U(1)_{\text{gauge}}$ with charges $q_1 = 4$, $q_2 = 1$. (NB: the label "gauge" is only a label. From the point of view of group theory, this is just $U(1)$.)

1. Write down the most generic renormalizable Lagrangian for the model.
2. The model has a trivial global symmetry and a non-trivial one. What are they?
3. Now include in the Lagrangian higher order terms up to mass dimension 6. What is the fate of global symmetries?

This problem is a simplified version of what happens in the SM as well. The baryon and the lepton numbers are *accidental symmetries* of the SM: they are not imposed from the start, but they are consequences of the renormalizability of the Lagrangian and of the charge (representation) assignment under the gauge group. If one includes non-renormalizable terms, interpreted as low-energy limit of some heavy new physics, the baryon and lepton numbers may be violated.

3rd Week: Abelian gauge symmetry and QED

Exercise 3.1: Chiral Symmetry

Consider a Dirac fermion ψ and the following transformations:

$$\psi \to e^{i\theta}\psi \quad \text{and} \quad \psi \to e^{i\theta\gamma_5}\psi \tag{3.1}$$

1. Find the transformation laws of $\bar{\psi}$. Hint: for the γ_5 transformation, use a Taylor expansion of the exponential.
2. Show that the Lagrangian of a massless, free Dirac fermion is invariant under Eq. (3.1)
3. Find the transformation for the left and right Weyl spinors under Eq. (3.1), demonstrating that $e^{i\theta\gamma_5}\psi$ is an axial transformation.
4. Is the Dirac mass term $m\bar{\psi}\psi$ invariant under (3.1)?.

Now consider a system of N fermions ψ_i that carry the same charge under $U(1)$, interacting with a neutral scalar ϕ:

$$\mathscr{L} = \bar{\psi}_i[i\partial\!\!\!/\delta_{ij} - m_{ij} - \lambda_{ij}\phi]\psi \tag{3.2}$$

5. For generic m_{ij}, λ_{ij} there is a U(1) symmetry. What is the symmetry group if $\lambda_{ij} \propto m_{ij}$?
6. Find a transformation for $\psi_{L,R}$, ϕ that imposes $m_{ij} = 0$ but allows $\lambda_{ij} \neq 0$. Explain why the symmetry must be chiral.

Exercise 3.2: Covariant Derivative & Gauge Fields

1. Show explicitly that the covariant derivatives $D_\mu\phi$ and $D\!\!\!/\psi$ transform as the field themselves under a gauge transformation.
2. Show that $F_{\mu\nu}$ is gauge invariant.
3. Show that $[D_\mu, D_\nu]\psi = igq\,F_{\mu\nu}\psi$. This relation can be used as an alternative definition of $F_{\mu\nu}$.
4. Show that $F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ is a total derivative, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor with $\epsilon^{0123} = +1$.

Exercise 3.3: Compton scattering

1. Draw the Feynman diagrams for the scattering process $e^-\gamma \to e^-\gamma$. At tree level there are two diagrams. In one, the exchanged momentum is $p_1^\mu + p_2^\mu$, in the other $p_2^\mu - p_4^\mu$. From the definition of the Mandestalm variables $(p_1+p_2)^2 = (p_3+p_4)^2 \equiv s$, $(p_1-p_3)^2 = (p_2-p_4)^2 \equiv t$ and $(p_1 - p_4)^2 = (p_2 - p_3)^2 \equiv u$, the two diagrams are said to be in the $s-$channel and in the $t-$channel, respectively.
2. Compute the matrix element.

3. Square the matrix element, sum over the final spin/polarization states, and average over the initial one. You can use the replacement

$$\sum_{\text{pols. } i} \epsilon_\mu^{i\,*} \epsilon_\nu^i \to -g_{\mu\nu} \,. \tag{3.3}$$

After some work you should find

$$\frac{1}{4} \sum_{\text{spins/pols.}} |\mathcal{M}|^2 = 2e^4 \left[\frac{p_{24}}{p_{12}} + \frac{p_{12}}{p_{24}} + 2m^2 \left(\frac{1}{p_{12}} - \frac{1}{p_{24}} \right) \right] , \tag{3.4}$$

where $p_{ij} = (p_i)_\mu (p_j)^\mu$. Hint: you will find traces of six gamma matrices, which can be simplified using $\gamma^\nu \not{p} \gamma_\nu = -2\not{p}$.

We want to evaluate the cross section in the lab frame, in which the electron is initially at rest.

4. Writing the momenta as

$$\begin{aligned} p_1 &= (\omega, 0, 0, \omega) & p_2 &= (m_e, 0, 0, 0) \\ p_4 &= (\omega', \omega' \sin\theta, 0, \omega' \cos\theta) & p_3 &= p_1 + p_2 - p_4 = (E', \vec{p}\,') \end{aligned} \tag{3.5}$$

show that $\omega' = \frac{\omega}{1 + \frac{\omega}{m_e}(1 - \cos\theta)}$, and thus, for the wavelength,

$$\Delta\lambda = \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m_e}(1 - \cos\theta) \tag{3.6}$$

which is Compton's formula for the wavelength shift. No QED is involved here: just relativistic kinematics.

5. Evaluate $d\Pi_{\text{LIPS}}$ in the lab frame, and use it to compute the cross section. You should obtain the *Klein-Nishina formula*

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2} \left(\frac{\omega'}{\omega} \right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right] . \tag{3.7}$$

6. Finally, go to the non-relativistic limit $m_e \gg \omega$. Obtain the Thompson scattering cross section for the classical scattering of light from a free electron:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2} [1 + \cos^2\theta] \,. \tag{3.8}$$

4^{th} Week: QED

Exercise 4.1: Anomalous magnetic moment

This problem concern the derivation of the (anomalous) magnetic moment of the electron. This is one of the real milestones of the development of QFT, and was decisive to convince people of the power of QED. We will work at the border between QFT and relativistic quantum mechanics, without discussing in details how to translate our result into the Schroedinger picture of QM. You can find more details on this point in the Lecture Notes by Ben Simons (Cambridge): https://www.tcm.phy.cam.ac.uk/ bds10/aqp.html

1. Start from the Dirac equation of relativistic quantum mechanics: $(i\not{\partial} - e\not{A} - m)\psi = 0$. Multiply it times $(i\not{\partial} - e\not{A} + m)$ on the left and obtain the equation

$$\left((i\partial_\mu - eA_\mu)^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\right)\psi = 0 \tag{4.1}$$

2. Compute explicitly the term $\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$ in terms of the $\vec{E}$ and $\vec{B}$ fields and show that this term leads to a magnetic dipole interaction $\vec{S}\cdot\vec{B}$.[1]

3. Derive the Hamiltonian by going to Fourier space, replacing $i\partial_t\psi = H\psi$ and $i\partial_i = p_i\psi$, and then realizing that, in the non-relativistic limit,

$$H\psi \approx (m + H_{\text{int}})\psi \qquad \text{and} \qquad (H - eA_0)^2\psi \approx [m^2 + 2m(H_{\text{int}} - eA_0)]\psi \tag{4.2}$$

   Now you should be able to read off from your Hamiltonian the magnetic dipole interaction

$$g\frac{e}{2m}\vec{B}\cdot\vec{\sigma}\,\psi \tag{4.3}$$

   with $g = 2$.

From the QM point of view, g is a parameter that could take any value. Explaining the experimental observation $g \approx 2$ was one of the great successes of the Dirac equation. $g = 2$ comes from the prefactor $e/2$ in front of $F_{\mu\nu}\sigma^{\mu\nu}$ in Eq. (4.1): were it g', we would have had $g = g'$. Thus, a useful way of finding corrections to $g = 2$, is to look for loop diagrams that have the same effect of an additional $F_{\mu\nu}\sigma^{\mu\nu}$ term. The relevant process to look at is the scattering of an electron off an off-shell, classical photon field A_μ:[2]

$$e^-(q_1)A_\mu(p) \to e^-(q_2)\,. \tag{4.4}$$

4. Start from the tree level matrix element, with the usual spinors $u, \bar{u}$ for the external electrons and writing A^μ_{cl} for the vector field. Using the Gordon identity, show that it can be written in the form

$$\mathcal{M} = A_{\mu\,\text{cl}}\mathcal{M}^\mu = \text{stuff } - \frac{e}{2m}i\bar{u}(q_2)\sigma^{\mu\nu}p_\nu u(q_1)A_{\mu\,\text{cl}} \tag{4.5}$$

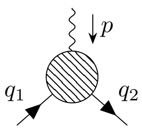
   The second term is the one responsible for $g = 2$ above.

[1] You will find an imaginary term proportional to the electric field: it does not represent an electric dipole, but it is just there because of Lorentz covariance. In a fixed reference frame you can simply set it to zero.

[2] Wait, what? A classical, off-shell field? Aren't all fields quantized? Aren't all external legs on-shell? You will understand this better (or simply get used to that) if you continue studying QFT!

5. Go to the one loop level. Draw the diagram responsible for the transition.

6. This 1-loop matrix element can be written as



$$= i\bar{u}(q_2)\Gamma^\mu u_(q_1)A_\mu(p) \tag{4.6}$$

   Without computing the loop diagram, find the most generic form of Γ^μ as a sum of four terms, each of them is a vector (including γ^μ) with index μ times a coefficient that depends only on the scalar products of the same four vectors. Using momentum conservation at the vertex, the fact that electrons are on-shell and the Dirac equation, show that the coefficient depend on p^2 only.

7. Now use the Ward and the Gordon identity to prove that $\mathcal{M}$ can be written in the form

$$i\mathcal{M} = i\bar{u}(q_2)\left[F_1(p^2)\gamma^\mu + F_2(p^2)F_{\mu\nu}\sigma^{\mu\nu}\right]u_(q_1)A_\mu(p) \tag{4.7}$$

The factor F_1 is responsible for the renormalization of the electric charge. The factor F_2 instead gives the *anomalous* magnetic moment of the electron, *ie* the deviation of g from 2:

$$g = 2 + 2F_2(0)\,. \tag{4.8}$$

$F_2(0)$ can be computed performing the loop integral. It turns out to be finite, and the result is $F_2(0) = \alpha/(2\pi)$, ie

$$g = 2 + \frac{\alpha}{\pi}\,. \tag{4.9}$$

5^{th} Week: QCD

Exercise 5.1: Non-Abelian Gauge Fields

Let us consider a theory invariant under a local $SU(N)$ symmetry:

1. Starting from the covariant derivative, show that, for any representation t^a_R of the gauge group, $\frac{i}{g}[D_\mu, D_\nu] = G^a_{\mu\nu} t^a_R$.
2. Find how the gauge fields must transform from the transformation law of the covariant derivative; what is the infinitesimal transformation?
3. Derive the transformation law of the field strength $G^a_{\mu\nu}$
4. Derive the Noether currents of the theory including fermions.

Exercise 5.2: Inclusive $e^+e^- \to \text{hadrons}$ cross section

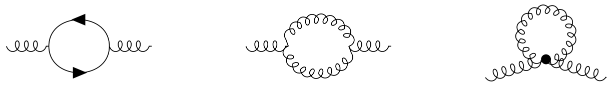
Starting from the Feynman rules, compute at tree level

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{5.1}$$

where $\sigma(e^+e^- \to \text{hadrons}) \approx \sigma(e^+e^- \to q\bar{q})$ at tree level. You can use the result for $\sigma(e^+e^- \to \mu^+\mu^-)$ derived in the lectures.

Exercise 5.3: Loop corrections and group representation matrices

Write down the corrections to the gluon propagator corresponding to the three Feynman diagrams



Do not compute the integrals, but take care of the necessary contractions of SU(3) generators.

6^{th} Week: Spontaneous symmetry breaking

Exercise 6.1: Goldstone boson scattering

Consider the Lagrangian of a complex scalar field:

$$\mathcal{L} = |\partial_\mu \phi|^2 + \mu^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 \tag{6.1}$$

We will consider two different parametrization of the fields after spontaneous symmetry breaking, and show that, although the Lagrangians are different, they describe the same physics.

1. Use the polar decomposition

$$\phi(x) = \left(v + \frac{\sigma(x)}{\sqrt{2}}\right) e^{i\pi(x)/F_\pi} \tag{6.2}$$

   with real σ, π and rewrite the Lagrangian. Show that, if v is chosen to minimize the potential, the linear term in σ cancels. Find the correct value of F_π and the masses of π and σ.

2. Now use the linear decomposition

$$\phi(x) = v + \frac{\chi_1 + i\chi_2}{\sqrt{2}}\,. \tag{6.3}$$

   Check that the normalization of the real fields χ_i is correct. Rewrite the Lagrangian and show that the masses are the same as above. This is the first check that the two theories are equivalent: the mass spectrum is the same.

3. Write the relation between σ, π and χ_1, χ_2 as a series. With a simple drawing on cartesian axes, identify the fields to show that they represent the same fluctuations at linear order.

4. Compute the Feynman rule of the $\pi\pi\sigma$ vertex in the polar theory. You can use the notes on Feynman diagrams that you find on Moodle.

5. Compute the Feynman rules of the χ_2^4 and $\chi_2^2\chi_1$ vertices in the linear theory.

6. Show that the matrix element for the $\pi\pi \to \pi\pi$ and $\chi_2\chi_2 \to \chi_2\chi_2$ scatterings are the same, once the external particles are taken on-shell. It is convenient to use Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2 \tag{6.4}$$

   and the relation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 = 4m_\pi^2 = 0\,. \tag{6.5}$$

This is a simple case of a more general theorem, that ensures the equivalence of the scattering matrix under a broad class of field redefinitions, as long as the field operators create the same particles from the vacuum. For more details you can look at Weinberg vol. 1, ch. 7.

Exercise 6.2: The linear sigma model

In this problem we will construct a phenomenological Lagrangian for the interactions of pions with nucleons in QCD. This model predates QCD itself, and was developed before the introduction of quarks. Nowadays, we understand it as an effective theory of strong interactions at low energies.

We consider a theory with a spontaneously broken global symmetry:

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B \to \mathrm{SU}(2)_V \times \mathrm{U}(1)_B\,. \tag{6.6}$$

There are two Weyl fermions for the left and right-handed nucleon doublets

$$N_L(2,1)_1 = \begin{pmatrix} p_L \\ n_L \end{pmatrix}, \qquad N_R(1,2)_1 = \begin{pmatrix} p_R \\ n_R \end{pmatrix}. \tag{6.7}$$

We introduce a scalar field Σ, that will describe into the Lagrangian 3 Goldstone bosons and a fourth scalar responsible for SSB.

1. Show that, to allow a renormalizable interaction term of Σ with N_L and N_R, Σ must transform as a 2 under $\mathrm{SU}(2)_L$ and as a $\bar{\mathbf{2}}$ under $\mathrm{SU}(2)_R$. Having two indices, you can think of Σ as a matrix $\Sigma_{a\bar{b}}$, where $\bar{b}$ is a $\mathrm{SU}(2)_R$ index. Write down the transformation law of Σ in matrix notation and an interaction term for $\Sigma N_L N_R$.

Σ as a 2×2 complex matrix has 8 real dof. We want to impose a reality condition that reduces the dof to 4. We first need to convince ourselves that the reality condition is not spoiled by transformations.

2. Show that $\tilde{\Sigma} \equiv \sigma^2 \Sigma^* \sigma^2$ transforms in the same representation of Σ. Use that $\sigma^2 U \sigma^2 = U^*$ for any $U \in \mathrm{SU}(2)$.

3. Expand Σ in the complex basis $\{\mathbb{1}, \sigma^a\}$ and impose the *reality* condition $\Sigma = \tilde{\Sigma}$. Conclude that it is consistent to assume

   $$\Sigma(x) = s(x)\mathbb{1} + i\pi^a(x)\frac{\sigma^a}{2} \tag{6.8}$$

   for *real* s, π^a.

4. Write down a Lagrangian for the Σ field, including mass and quartic terms, using traces to obtain invariant terms. Remember that the mass term in $\mathcal{L}$ must have a + sign to induce spontaneous symmetry breaking. You should have two quartic terms: using the decomposition in Eq. 6.8 show that they are equivalent, and you can keep one of them by rescaling the coupling λ.

5. Write the field as $\Sigma(x) \sim (F_\pi + \sigma(x))\mathbb{1} + i\pi^a(x)\sigma^a$ and write the Lagrangian (you should find the correct normalization). Rewrite the Lagrangian in terms of these fields. At the very end you should find, neglecting nucleons,[3]

   $$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi^a)^2 + \frac{1}{2}m^2\sigma^2 - \lambda F_\pi \sigma(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \tag{6.9}$$

   Hint: start by simplifying $\Sigma^\dagger\Sigma$. You should have found two quartic terms: expanding Σ you can show easily that they are related to each other

[3] This is called the *linear sigma-model*: the symmetry acts linearly on Σ, and the order parameter is called σ.

6. Put nucleons back. Expand the Lagrangian and extract the mass of all the particles. You should find the *Goldberger-Treiman* relation:

$$m_N = g_{\pi NN} F_\pi \,, \tag{6.10}$$

where $g_{\pi NN}$ is the coupling constant of the $\bar{N}_L \Sigma N_R$ term.

7^{th} Week: Spontaneous symmetry breaking II

Exercise 7.1: Chiral Lagrangian

We build on last week's problem on the linear sigma model.

1. Expand Σ as
$$\Sigma(x) = \frac{v+\sigma(x)}{\sqrt{2}} \exp\left(i\frac{\pi^a(x)\sigma^a}{2F_\pi}\right) \equiv \frac{v+\sigma(x)}{\sqrt{2}} U(x) \tag{7.1}$$
Check the kinetic terms to find the relation between F_π and v.

2. Consider the decoupling limit $\mu \to \infty, \lambda \to \infty$, with F_π fixed. What happens to the masses of the nucleons? And to σ?

3. Write down a Lagrangian for the Goldstones in terms of the matrix U.

4. Quark masses break the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry explicitly. Write the QCD mass term as
$$\mathcal{L}_m \bar{q} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} q \tag{7.2}$$
The symmetry can be restored if we pretend for a moment that M, instead of being a constant matrix, transforms under the symmetry group. How should M transform to make $\mathcal{L}_m$ invariant?[4]

5. Using M with its spurious transformation property, write down a new term in the Lagrangian involving M and U, with an arbitrary coefficient V^3 in front.

6. Expand the exponential in U. You can find a vacuum energy piece and a mass term for the pions. You can fix the coefficient V^3 by imposing that the vacuum energy equals that obtained by $\mathcal{L}_m$ after symmetry breaking. Now compute the pion masses. You should obtain the *Gell-Mann–Oakes–Renner* relation:
$$m_\pi^2 = \frac{V^3}{F_\pi^2}(m_u + m_d)\,. \tag{7.3}$$

Exercise 7.2: Spontaneous breaking of SU(5) and Grand Unification

1. A simple way of constructing the generators T^a of (the fundamental rep of) $\mathrm{SU}(N)$ is the following:

   - The first $(N-1)^2-1$ generators are obtained from those of $\mathrm{SU}(N-1)$, adding the last row and the last column of 0's. These are the generators of the $\mathrm{SU}(N-1)$ subgroup.
   - $N-1$ generators are built with 1 on the Ni and iN entries, similar to the σ_1 matrix. A factor 1/2 in front ensures the correct normalization $\mathrm{Tr}(\mathrm{T^a T^b}) = \frac{1}{2}\delta^{\mathrm{ab}}$
   - Another $N-1$ generators are built with i and $-i$ on the Ni and iN entries, similar to the σ_2 matrix.
   - The last matrix is
$$T^{N^2-1} = \mathrm{const} \times \mathrm{diag}(1, 1, \ldots, -(N-1)) \tag{7.4}$$

[4] From the point of view of the symmetry, M is called a spurion: if it could be transformed, the Lagrangian would be invariant.

Construct in this way a set of generators for SU(5), and find the correct normalization of the diagonal ones

2. Consider the Lagrangian for a scalar field ϕ^a, transforming in the *adjoint* representation. Is it consistent to take ϕ as real?

3. Show that the kinetic term can be rewritten as

$$\frac{1}{2}(D_\mu \phi^a)^2 = \text{Tr}[(D_\mu \Phi)^\dagger (D^\mu \Phi)]\,, \tag{7.5}$$

where $\Phi = \phi^a T^a$, T^a being the generators in the fundamental rep introduced above. Rewrite $(D_\mu \Phi)$ using the commutator $[\mathbb{A}_\mu, \Phi]$, where $\mathbb{A}_\mu = A^a_\mu T^a$.

4. The gauge symmetry is broken by a vev

$$\langle \Phi \rangle = A \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \tag{7.6}$$

What is the residual symmetry after SSB? How many massive and massless vectors does the model predict?

5. From the kinetic term written in matrix form, deduce the spectrum of the massive vectors.

6. Determine the unbroken symmetry and spectrum of vector bosons when the symmetry is broken by a vev

$$\langle \Phi \rangle = B \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{7.7}$$

8^{th} Week: The Standard Model

Exercise 8.1: Explicit calculations

1. Determine the transformation law of $\tilde{\phi} = i\sigma_2\phi$ under the SM gauge symmetry, and argue that it can be used to form a gauge invariant Yukawa for up quarks.

2. Starting from the covariant derivative $D_\mu = \partial_\mu - igW^3_\mu T^3 - ig' B_\mu Y \mathbb{1}$, derive the interactions of A_μ and Z_μ with generic field representations (do not specify T^3, Y) and with SM fermions.

3. Derive the Noether currents J^Y_μ, J^3_μ, J^{em}_μ.

4. Derive the interactions of $W^\pm$ with SM fermions in the flavour basis and in the mass basis. Write this piece of Lagrangian in the form

$$\mathcal{L} = \frac{e}{\sqrt{2}\sin\theta_W}(W^\mu_+ J^+_\mu + W^\mu_- J^-_\mu)\,. \tag{8.1}$$

   and identify the currents J^+_μ, J^-_μ.

Exercise 8.2: * Fermi theory and muon decay

Fermi theory is the best description of weak interactions at low energies, like those relevant for the beta decay nuclei or for muon decay.

1. Write down the tree level matrix element for the process $\mu^- \to e^-\nu_\mu\bar{\nu}_e$. Expand the result for $p^2 \ll m_W^2$ at the first non-trivial order.

2. Obtain the same result with the 4 fermion Lagrangian

$$\mathcal{L}_{4F} = \frac{4G_F}{\sqrt{2}} J^-_\mu J^{+\mu}\,. \tag{8.2}$$

   You will need to derive the Feynman rule for this vertex. What is the value of G_F as a function of m_W and as a function of v?

3. Square the matrix element, sum over the final spin states and average over the initial ones.

4. Consider the integral

$$I^{\mu\nu} \equiv \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2} q_1^\mu q_2^\nu \delta^{(4)}(q - q_1 - q_2)\,. \tag{8.3}$$

   $I^{\mu\nu}$ depends on q^μ only, and is symmetric in μ, ν. Thus it cn be written as $I^{\mu\nu} = g^{\mu\nu}A(q^2) + q^\mu q^\nu B(q^2)$. Show that

$$I^{\mu\nu} = \frac{\pi}{6}(g^{\mu\nu}q^2 + 2q^\mu q^\nu) \tag{8.4}$$

5. The rate for a three body decay $\mu^-(p) \to e^-(k)\nu_\mu(q_1)\bar{\nu}_e(q_2)$ is

$$d\Gamma = \frac{1}{2m_\mu}|\overline{\mathcal{M}^2}|(2\pi)^4\delta^{(4)}(p - k - q_1 - q_2)\frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}\frac{d^3k}{(2\pi)^3 2E_e} \tag{8.5}$$

   Using this and the above result for $I^{\mu\nu}$ with $q = p - k$, obtain the electron spectrum $d\Gamma/dE_e$.

6. Neglecting the electron mass m_e, find the integration limits for E_e and compute the total width:
$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{8.6}$$
7. From the experimental values $\tau_\mu \approx 2.2 \times 10^{-6}\,\mathrm{s}$ and $m_\mu \approx 105.7\,\mathrm{MeV}$ obtain the value of G_F.

Exercise 8.3: Phases of the CKM matrix

We have seen that there is only one physical phase in the quark sector of the Standard Model. In the mass basis for the quark fields and in the standard parametrization of V_{CKM}, this is the angle δ. Generalize this result to find the number of physical phases with N flavour generations:

$$\#(\text{physical phases}) = \frac{(N-1)(N-2)}{2}\,. \tag{8.7}$$

This result, obtained by Kobayashi and Maskawa in 1973, was the first indication of the existence of a third quark family to explain CP violation (which was known since 1964).

9^{th} Week: Phenomenology of the SM

Exercise 9.1: * FCNC at tree level with three quarks [1]

Before the charm quark was discovered, three quarks were known: u, d, s. A model of this kind leads to large FCNC already at three level, a fact that was already then in tension with data, as we will see in this exercise.

Consider a model with the SM gauge symmetry, the usual Higgs doublet $\phi(1,2)_{1/2}$ with $\langle\phi\rangle = (0, v/\sqrt{2})$, two lepton families with the usual quantum numbers, and only three quark fields organised as follows:

$$Q_L(3,2)_{1/6} \quad s_L(3,1)_{-1/3} \quad u_R(3,1)_{2/3} \quad d_R(3,1)_{-1/3} \quad s_R(3,1)_{-1/3} \tag{9.1}$$

1. In the interaction basis, and only for the quarks, write down
   (a) The covariant derivative of the quark fields
   (b) The charged current weak interactions
   (c) The Yukawa interactions of ϕ^0
   (d) The bare mass terms
   (e) The quark mass matrix after SSB.
2. How many physical flavour parameters are there? Separate them into masses, angles and phases.
3. Based on symmetries and what we have seen in class, are there photon- or gluon-mediated FCNCs?
4. Write down the Higgs interactions with quarks in the mass basis. Are there FCNCs at tree level? In which sector?
5. Write down the W boson interactions with quark mass eigenstates.
6. In the SM, the rate of $\bar{s} \to \bar{u}\mu^+\nu_\mu$ is used to determine $V_{us} \approx 0.2243$. Obtain the value of the mixing angle in $\mathcal{L}_W$ using this result.
7. Write down the Z interactions with quarks in both the interaction and the mass basis. Are there FCNCs at tree level? In which sector?

Exercise 9.2: * FCNC at tree level in two Higgs doublets model [1]

Consider the two Higgs doublet model (2HDM). This is an extension of the SM with two scalar multiples instead of the usual one:

$$\phi_1(1,2)_{1/2}\,, \qquad \phi_2(1,2)_{1/2}\,. \tag{9.2}$$

Suppose the potential is such that both fields get a real vev along their down component:

$$\langle\phi_1\rangle = \begin{pmatrix} 0 \\ v_1/\sqrt{2} \end{pmatrix}, \qquad \langle\phi_2\rangle = \begin{pmatrix} 0 \\ v_2/\sqrt{2} \end{pmatrix}. \tag{9.3}$$

Define the angle β as

$$\tan\beta = v_2/v_1\,. \tag{9.4}$$

We make the simplifying assumption that CP is conserved (thus, all couplings are real).

1. Consider the kinetic term of the two scalars. Show that the W and Z masses are the same as in the SM, with $v^2 = v_1^2 + v_2^2$.

2. Write down the most general Yukawa Lagrangian for the quarks.

3. Find the mass of the quarks in terms of the Yukawa couplings and of the vevs v_1, v_2.

4. Derive the coupling of the Z and W bosons to quark mass eigenstates. You should find that these couplings are the same as in the SM.

5. We expand the two doublets around their vevs as

$$\phi_1 = \begin{pmatrix} \eta_1^+ \\ (v_1 + h_1 + i\chi_1)/\sqrt{2} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \eta_2^+ \\ (v_2 + h_2 + i\chi_2)/\sqrt{2} \end{pmatrix}, \tag{9.5}$$

where η_i^+ are charged (complex) scalars, h_i (CP-even) and χ_i (CP-odd) are real scalars. The 5 mass eigenstates of the scalar sector are

$$H^\pm = -s_\beta\, \eta_1^\pm + c_\beta\, \eta_2^\pm \tag{9.6}$$
$$A = -s_\beta\, \chi_1 + c_\beta\, \chi_2 \tag{9.7}$$
$$H = c_\alpha\, h_1 + s_\alpha\, h_2 \tag{9.8}$$
$$h = -s_\alpha\, h_1 + c_\alpha\, h_2 \tag{9.9}$$

where α is some angle that we don't specify here and depends on the exact form of the scalar potential. Find the Yukawa couplings of A, H, h to quarks in the mass basis. Show that these are, in general, non diagonal. This implies the existence of tree level FCNCs.

6. Impose the additional $\mathbb{Z}_2$ symmetry under which

$$\phi_1 \to \phi_1 \tag{9.10}$$
$$\phi_2 \to -\phi_2 \tag{9.11}$$
$$Q_L \to Q_L \tag{9.12}$$
$$d_R \to d_R \tag{9.13}$$
$$u_R \to -u_R \tag{9.14}$$

Show that there are no Higgs mediated FCNCs. This pattern is usually called *Type II 2HDM*, and is assumed in many well motivated extensions of the SM such as the *minimal supersymmetric SM* (MSSM).

Exercise 9.3: * Decay width of the Z boson

1. Compute the branching ratios of the Z boson into lepton and into quark pairs. You don't need to compute the decay widths.

2. Compute the partial decay widths and the total one.

Exercise 9.4: Decay width of the W boson

1. Compute the branching ratios of the W boson into fermion anti-fermion pairs, and the branching ratio in each channel. Why is $\text{BR}(W^- \to \text{hadrons}) \approx 2 \times \text{BR}(W^- \to \text{leptons})$? You don't need to compute the decay widths.

2. Compute the partial decay widths and the total one.

Exercise 9.5: * Decay width of the t quark

Compute the decay width of the top quark into W^+b. Separate the calculation into two contributions, for transverse and longitudinal W.

References

[1] Y. Grossman and Y. Nir, *The Standard Model: From Fundamental Symmetries to Experimental Tests*. Princeton University Press, 10, 2023.