Lesson 7 - 10/3/2026

next week

Tuesday March 17^{th}, 9 - 11 am room Z building G
Thursday March 19^{th}, 6-7 pm room 5B buil. H2/bis
Friday " 20^{th} 9-11 am room 55 buil H2/bis

· signed and complex measures. (ch. 19 - HS)

def. let Ω set, $\mathcal{A}$ σ-algebra in $\mathcal{P}(\Omega)$
let $\nu : \mathcal{A} \to]-\infty, +\infty]$ (or $\nu : \mathcal{A} \to [-\infty, +\infty[$)

ν will be a <u>signed measure</u> if

i) $\nu(\emptyset) =$

ii) ν countably additive

<u>rem.</u> ν countably additive means
let $(A_n)_n$ sequence in $\mathcal{A}$ s.t. $A_j \cap A_k = \emptyset$ if $j \neq k$
then if $\nu(\bigcup_n A_n) < +\infty$
then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(\bigcup_n A_n)$

if $\nu(\bigcup_n A_n) = +\infty$

then, denoting $B_n = \begin{cases} A_n & \text{if } \nu(A_n) \geq 0 \\ \emptyset & \text{if } \nu(A_n) < 0 \end{cases}$ $C_n = \begin{cases} A_n & \text{if } \nu(A_n) < 0 \\ \emptyset & \text{if } \nu(A_n) \geq 0 \end{cases}$

then $\sum_n \nu(B_n) = +\infty$ and $\sum_n \nu(C_n) < +\infty$

<u>similarly</u>

def $(\Omega, \mathcal{A})$ measurable space
$\nu : \mathcal{A} \to \mathbb{C}$
ν is a complex measure if
i) $\nu(\emptyset) = 0$
ii) ν is countably additive

<u>rem.</u> in this case let $(A_n)_n$ in $\mathcal{A}$ pairwise disjoint sets

then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(\bigcup_n A_n)$

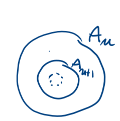
with these definitions it is possible to prove

<u>Th.</u> let ν a signed or complex measure on $\mathcal{A}$
then

i) if $E, F \in \mathcal{A}$ with $E \subseteq F$
and $|\nu(F)| < +\infty$ then $|\nu(E)| < +\infty$

ii) let $(A_n)_n$ in $\mathcal{A}$ and $\forall n$, $A_n \subseteq A_{n+1}$
then $\nu(\bigcup_n A_n) = \lim_n \nu(A_n)$

iii) let $(A_n)_n$ in $\mathcal{A}$ and $\forall n$, $A_n \supseteq A_{n+1}$
then if $|\nu(A_0)| < +\infty$
$\lim_n \nu(A_n) = \nu(\bigcap_n A_n)$



Hahn's decomposition theorem

def. let ν be a signed measure (on $\mathcal{A}$ of $\mathcal{P}(\Omega)$),
the couple (P, N) with $P, N \in \mathcal{A}$
is called <u>Hahn's decomposition</u> of ν if

i) $P \cap N = \emptyset$, $P \cup N = \Omega$

ii) $\forall A \in \mathcal{A}$, $\nu(P \cap A) \geq 0$ — P is a non negative set
$\forall A \in \mathcal{A}$, $\nu(N \cap A) \leq 0$ — N is a non positive set

(idea $f \in L^+_\lambda(\Omega)$
define $\nu(A) = \int_A f \, d\lambda$ (ν is a signed measure)
$P = \{x \in \Omega : f(x) \geq 0\}$
$N = \{x \in \Omega : f(x) < 0\}$)

Theorem let ν be a signed measure
then there exists a Hahn's decomposition
and this is <u>unique</u> neglecting sets of measure 0.

proof
lemma 1. take ν a signed measure
take $E \in \mathcal{A}$ with $-\infty < \nu(E) < +\infty$.
then $\forall \varepsilon > 0$, $\exists E_\varepsilon \in \mathcal{A}$ s.t. $E_\varepsilon \subseteq E$, $\nu(E_\varepsilon) \geq \nu(E)$
and $\forall A \in \mathcal{A}$ if $A \subseteq E_\varepsilon$ then $\nu(A) \geq -\varepsilon$.

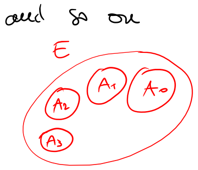
proof. I take $E \in \mathcal{A}$ s.t. $-\infty < \nu(E) < +\infty$
by contradiction
$\exists \varepsilon_0 > 0 : \forall F \in \mathcal{A}$
if $F \subseteq E$ and $\nu(F) \geq \nu(E)$
then $\exists A_0 \in \mathcal{A}$ s.t. $A_0 \subseteq F$ and $\nu(A_0) < -\varepsilon_0$

I take $F = E$
I have $E \subseteq E$, $\nu(E) \geq \nu(E)$
so that $\exists A_0 \in \mathcal{A}$: $A_0 \subseteq E$ and $\nu(A_0) < -\varepsilon_0$

now I take $F = E \setminus A_0$
I have $E \setminus A_0 \subseteq E$ and $\nu(E \setminus A_0) = \nu(E) \underbrace{- \nu(A_0)}_{> \varepsilon_0} \geq \nu(E)$
so $\exists A_1 \in \mathcal{A}$: $A_1 \subseteq E \setminus A_0$ and $\nu(A_1) < -\varepsilon_0$

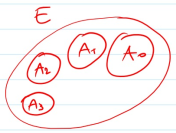
then I take $F = E \setminus (A_0 \cup A_1)$ and I go on
I obtain $\exists A_2 \in \mathcal{A}$, $A_2 \subseteq E \setminus (A_0 \cup A_1)$ with $\nu(A_2) < -\varepsilon_0$

and so on



I obtain $(A_n)_n$ in E pairwise disjoint s.t. $\nu(A_n) < -\varepsilon_0$

and so on



I obtain $(A_n)_n$ in E
pairwise disjoint
s.t. $\nu(A_n) < -\varepsilon_0$

so that $(\bigcup_n A_n) \subseteq E$
but $\sum_n \nu(A_n) = -\infty$ } impossible since $-\infty < \nu(E) < +\infty$

<u>Lemma 2</u> let $E \in \mathcal{A}$, with $-\infty < \nu(E) < +\infty$

$\exists F \in \mathcal{A}$ s.t. $F \subseteq E$, $\nu(F) \geq \nu(E)$
and $\forall A \in \mathcal{A}$, $\nu(F \cap A) \geq 0$

(remark that F will be a non negative set)

<u>proof.</u> I have E with $-\infty < \nu(E) < +\infty$

I fix $\varepsilon = 1$ I apply Lemma I

$\exists E_1 \in \mathcal{A}$ st. $E_1 \subseteq E$, $\nu(E_1) \geq \nu(E)$
and $\forall A \in \mathcal{A}$, if $A \subseteq E_1$ then
$\nu(A) > -1$

now I fix $\varepsilon = \frac{1}{2}$ I apply Lemma 1 to E_1
(remark that $E_1 \subseteq E$
so that $-\infty < \nu(E_1) < +\infty$)

$\exists E_2 \in \mathcal{A}$ s.t $E_2 \subseteq E_1 \subseteq E$, $\nu(E_2) \geq \nu(E_1) \geq \nu(E)$
and $\forall A \in \mathcal{A}$, if $A \subseteq E_2$ then
$\nu(A) \geq -\frac{1}{2}$

I fix $\varepsilon = \frac{1}{3}$ I apply Lemma to E_2

$\exists E_3 \in \mathcal{A}$ st. $E_3 \subseteq E_2 \subseteq E_1 \subseteq E$
$\nu(E_3) \geq \nu(E_2) \geq \nu(E_1) \geq \nu(E)$
and $\forall A \in \mathcal{A}$ if $A \subseteq E_3$ then $\nu(A) > -\frac{1}{3}$

and so on

$\exists E_n \in \mathcal{A}$ s.t. $E_n \subseteq E_{n-1} \subseteq E$, $\nu(E_n) \geq \nu(E)$
$\forall A \in \mathcal{A}$, if $A \subseteq E_n$ then $\nu(A) \geq -\frac{1}{n}$

I take $F = \bigcap_n E_n$
I have $F \subseteq E$, $\nu(F) = \lim_n \underbrace{\nu(E_n)}_{\geq \nu(E)} \Rightarrow \nu(F) \geq \nu(E)$

finally $A \in \mathcal{A}$

if $A \subseteq F$ then $A \subseteq E_n$ $\forall n$
then $\nu(A) \geq -\frac{1}{n}$ $\forall n$
$\Rightarrow \nu(A) \geq 0$ <u>QED</u>

proof (of Hahn's dec. th.)

suppose $\nu : \mathcal{F} \to [-\infty, +\infty[$

let $\alpha = \sup_{A \in \mathcal{F}} \nu(A)$ (in principle it can be $\alpha = +\infty$)

(if $\alpha = -\infty \Rightarrow \forall A \in \mathcal{F}\ \nu(A) = -\infty$ trivial situation)

let $\alpha > -\infty$ (if $\alpha \le 0$ trivial situation, interesting case $\alpha > 0$)

consider $(E_n)_n$ in $\mathcal{F}$ s.t. $\lim_n \nu(E_n) = \alpha$

remark that, it is not restrictive $-\infty < \nu(E_n) < +\infty$

I apply lemma 2 to E_n

$\exists F_n \in \mathcal{F}$ s.t. $F_n \subseteq E_n$, $\nu(F_n) \geq \nu(E_n)$

and $\forall A \in \mathcal{F}$, $\nu(A \cap F_n) \geq 0$

remark that $\lim_n \nu(F_n) = \alpha$ (since $\lim_n \nu(E_n) = \alpha$)

define $G_n = F_0 \cup F_1 \cup \dots \cup F_n$

now $(G_n)_n$ is increasing sequence

$\lim_n \nu(G_n) = \alpha$ ($\nu(G_n) \geq \nu(F_n)$ since $F_0, F_1, \dots F_n$ are non negative)

but $\lim_n \nu(G_n) = \nu(\bigcup_n G_n)$

so that $\alpha = \nu(\bigcup_n G_n)$ and $\alpha < +\infty$

finally $\forall A \in \mathcal{F}$, if $A \subseteq \bigcup_n G_n$ then $\nu(A) \geq 0$

$A \subseteq \bigcup_n G_n \Rightarrow A = \bigcup_n (G_n \cap A)$ and $\nu(G_n \cap A) \geq 0$ $\forall n$

I set $P = \bigcup_n G_n$

I have already seen that P is non negative

now I prove that $\Omega \setminus P$ is non positive

(I will have that $(P, \Omega \setminus P)$ is the wanted decomp.)

suppose by cont. that $\Omega \setminus P$ is not non positive

($\Omega \setminus P$ non positive means $\forall B \subseteq \Omega \setminus P$ $\nu(B) \leq 0$)

$\left(\Omega \setminus P \text{ non positive means } \begin{array}{l} \forall B \subseteq \Omega \setminus P \\ \nu(B) \le 0 \end{array} \right)$

$\exists B_0 \subseteq \Omega \setminus P$ s.t. $\nu(B_0) > 0$

so $\nu(P \cup B_0) = \underset{\alpha}{\nu(P)} + \underbrace{\nu(B_0)}_{>0} > \alpha$ impossible

Ex. prove the uniqueness

(P_1, N_1), (P_2, N_2) Hahn's dec.

$\nu(P_1 \cap N_2)$ ≥ 0 (P_1 non negative), ≤ 0 (N_2 non positive)

$\Rightarrow \nu(P_1 \cap N_2) = 0$ similarly $\nu(P_2 \cap N_1) = 0$

$\nu(P_1 \setminus P_2) = \nu(P_2 \setminus P_1) = \nu(N_1 \setminus N_2) = \nu(N_2 \setminus N_1) = 0$

Summary

let ν be a signed measure (on $\mathcal{A}$ of $\mathcal{P}(\Omega)$)

$\exists (P, N)$ Hahn's decomp.

define $\nu^+ : \mathcal{A} \longrightarrow [0, +\infty]$

$\nu^- : \mathcal{A} \longrightarrow [0, +\infty]$

$\nu^+(A) = \nu(A \cap P)$

$\nu^-(A) = -\nu(A \cap N)$

ν^+, ν^- are positive measures

ν^+ positive variation of ν

ν^- negative variation of ν

$|\nu| = \nu^+ + \nu^-$ positive measure the total variation of ν

Ex

λ positive measure

$f \in L^1_\lambda$

$\nu_f(A) = \int_A f \, d\lambda$ $\quad f = f^+ - f^-$ $\quad f^+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$

$\nu_f^+(A) = \int_A f^+$, $\nu_f^-(A) = \int_A f^-$

$|\nu_f|(A) = \int_A |f|$ $\quad (f^+ + f^- = |f|)$

Characterization of the total variation of a signed measure

Th. let ν be a signed measure

define on $\mathcal{A}$ $\mu(E) = \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| : n \in \mathbb{N} \setminus \{0\},\ E_1, E_2, \dots E_n \in \mathcal{A} \text{ pairwise disjoint sets and } \bigcup_{j=1}^{n} E_j = E \right\}$

Th. let ν be a signed measure

define $\mu(E) = \sup\{ \sum_{j=1}^{n} |\nu(E_j)| : n \in \mathbb{N}\setminus\{0\},\ E_1, E_2, \dots E_n \in \mathcal{F}$ pairwise disjoint sets and $\bigcup_{j=1}^{n} E_j = E \}$

on $\mathcal{F}$

Then $\mu = |\nu|$

proof. first I prove that $\mu(E) \le |\nu|(E)$

take $E \in \mathcal{F}$

consider $E = \bigcup_{j=1}^{n} E_j$ with $E_j \cap E_k = \emptyset$ if $j \ne k$

then $\forall_j \quad |\nu(E_j)| \le |\nu|(E_j)$

by Hahn's $\quad \nu(E_j) = \nu^+(E_j) - \nu^-(E_j)$

$$|\nu(E_j)| = |\nu^+(E_j) - \nu^-(E_j)| \le |\nu^+(E_j)| + |\nu^-(E_j)| = \nu^+(E_j) + \nu^-(E_j) = |\nu|(E_j)$$

so that $\sum_j |\nu(E_j)| \le \sum_j |\nu|(E_j) = |\nu|(E)$

$\forall (E_j)_j \implies \mu(E) \le |\nu|(E)$

I prove the converse

let $E \in \mathcal{F}$ I take as partition

$E = (E \cap P) \cup (E \cap N)$

(where (P, N) is the Hahn's dec)

then $\underbrace{|\nu(E \cap P)| + |\nu(E \cap N)|}_{= |\nu|(E)} \le \sup\{\ \} = \mu(E)$

$|\nu(E \cap P)| = \nu^+(E)$, $|\nu(E \cap N)| = \nu^-(E)$

$|\nu| \le \mu$

QED

and what about complex measures?

Th let ν be a complex measure

then defining

$$\mu(E) = \sup\{ \sum_{j=1}^{n} |\nu(E_j)| : n \in \mathbb{N}\setminus\{0\},\ E = \bigcup_{j=1}^{n} E_j \text{ pairwise disjoint} \}$$

μ is a positive measure, I call it the total variation of ν

μ is also finite.

the proof that μ is a measure is on the ν notes

the proof that μ is a finite measure is in the book of Rudin

Conclusion

Let ν be a complex measure

$\nu = \mathrm{Re}\,\nu + i\, \mathrm{Im}\,\nu$ $\qquad$ $\mathrm{Re}\,\nu$, $\mathrm{Im}\,\nu$ signed measure

so that

$$\underbrace{\nu = \mathrm{Re}\,\nu^+ - \mathrm{Re}\,\nu^- + i\left(\mathrm{Im}\,\nu^+ - \mathrm{Im}\,\nu^-\right)}_{\text{the Jordan's decomposition of } \nu}$$

while $\underbrace{|\nu|}_{\text{total variation}}(E) = \sup\left\{ \sum_{j=1}^{n} |\nu(E_j)| \;\middle|\; \bigcup_{j=1}^{n} E_j = E \text{ pairwise disjoint} \right\}$.

$$\left(|\mathrm{Re}\,\nu|, \; |\mathrm{Im}\,\nu| \le |\nu| \quad , \quad |\nu| \le |\mathrm{Re}\,\nu| + |\mathrm{Im}\,\nu| \right)$$

H.S.