Some Results in Riemannian Geometry part 0

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Differentiable Manifolds - Def. I

Definition

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3) the transition maps

$$\mathbf{x}_\alpha \circ \mathbf{x}_\beta^{-1} : \mathbf{x}_\beta(V_\alpha \cap V_\beta) \to \mathbf{x}_\alpha(V_\alpha \cap V_\beta)$$

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$(V_\alpha, \mathbf{x}_\alpha)$ is called a (local) chart or local parametrization of M.
A family $\{(V_\alpha, \mathbf{x}_\alpha\}_\alpha$ which satisfies 1), 2) and 3) is called a differentiable structure on M.

Differentiable mappings

We say that a function $f : M \to \mathbb{R}$ is *differentiable at* $p \in M$ if, given any local chart $(V_\alpha, \mathbf{x}_\alpha)$ of M with $\mathbf{x}_\alpha(p) \in V_\alpha$, it turns out that $f \circ \mathbf{x}_\alpha^{-1}$ is differentiale at $\mathbf{x}_\alpha^{-1}(p)$.

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Definition

Given two differentiable manifolds M, M' of dimension n, n' respectively and a mapping $F : M \to M'$, we say that F is *differentiable at* $p \in M$ if, given a parametrization $(V'_\beta, \mathbf{y}_\beta)$ of M' at $F(p)$ there exists a parametrization $(V_\alpha, \mathbf{x}_\alpha)$ of M at p such that $F(\mathbf{x}_\alpha^{-1}(V_\alpha)) \subset \mathbf{y}_\beta^{-1}(V'_\beta)$ and

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Local basis:

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\} \Bigg|_p.$$

Differentiable Manifolds - Def. II

Definition

A differentiable manifold M of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ such that
1) $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$
2) for any α, β with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \varnothing$, the sets $\mathbf{x}_\beta^{-1}(W)$ and $\mathbf{x}_\alpha^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.
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Topology on M

Definition

If M is a differentiable manifold of dimension n, a set $A \subset M$ is defined to be open in M if and only if $\mathbf{x}_\alpha^{-1}(A \cap \mathbf{x}_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for any α.

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Remark

The topology is defined in such a way that the sets $\mathbf{x}_\alpha(U_\alpha)$ are open and the mappings $\mathbf{x}_\alpha$ are continuous.

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We say that a function $f : M \to \mathbb{R}$ is *differentiable at p* if, given any local chart $(U_\alpha, \mathbf{x}_\alpha)$ of M with $p \in V_\alpha$, it turns out that $f \circ \mathbf{x}_\alpha$ is differentiale at $\mathbf{x}_\alpha(p)$.

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If a differentiable mapping is also invertible with inverse also differentiable, we say that the mapping is a *diffeomorphism.*

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In a local parametrization (we omit the label α), let $f \in \mathcal{D}_p$ and $\gamma(t) = \mathbf{x}(x_1(t), \ldots, x_n(t))$ with $t \mapsto (x_1(t), \ldots, x_n(t))$ a curve in $U \subset \mathbb{R}^n$.

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$$\frac{d}{dt}(f \circ \gamma)\Big|_{t=0} = \frac{d}{dt} f(x_1(t), \ldots, x_n(t))\Big|_{t=0} = \sum_{j=1}^{n} x_j'(0) \frac{\partial f}{\partial x_j}$$

or, since $\gamma'(0)f = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$,

$$\gamma'(0) = \sum_{j=1}^{n} x_j'(0) \frac{\partial}{\partial x_j}\Big|_O.$$

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Remark

If $F : M \to M'$ is a differentiable map, then

$$dF_p : T_pM \to T_{F(p)}M'$$

(the differential *of F at p) is defined as follows: if $v = \gamma'(0)$, then $dF(v) = \eta'(0)$, where $\eta = F \circ \gamma$.*

Immersions and Embeddings

Let M and N be two differentiable manifolds.

Definition

A differentiable mapping $\Phi : M \to N$ is an *immersion* if $\forall p \in M$ $d\Phi_p$ is injective.
If, in addition, Φ is a homeomorphism of M onto $\Phi(M) \subseteq N$, we say that Φ is an *embedding*.

Vector Fields

Definition

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The set of all differentiable vector fields will be denoted by $\Gamma(TM)$.

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Let $X, Y \in \Gamma(TM)$, then

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Properties of the Lie bracket of vector fields X, Y, Z
1) $[X, Y] = -[Y, X]$
2) if $f, g \in \mathcal{D}(M)$, then

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$$

3)

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

(Jacobi identity)

Distributions

Definition

Given a differentiable manifold M of dimension n, a *distribution* on M of dimension k with $1 \leq k \leq n$ is a (smooth) choice of a subspace of dimension k in T_pM, i.e. a differentiable mapping

$$\Delta_k : M \to TM \quad p \mapsto (p, \Delta_k(p) \subseteq TpM),$$

with $\Delta_k(p)$ a linear subspace of dimension k in T_pM.

Distributions

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A distribution Δ_k on M of dimension k is *involutive* if it is closed with respect to Lie bracket operation, i.e. $X, Y \in \Delta_k \implies [X, Y] \in \Delta_k$.

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Definition

Given distribution Δ_k on M of dimension k, a differentiable submanifold N of M ($i : N \hookrightarrow M$, i immersion) is said *integral* of Δ_k if $\forall p \in N$ $di_p(T_pN) \simeq \Delta_k(p)$.

Frobenius Theorem

Theorem (Frobenius)

Given distribution Δ_k on M of dimension k, there exists a differentiable submanifold N of M integral of Δ_k if and only if Δ_k is involutive.

Something more on the topology of a differentiable manifold

From now on we'll always assume that the topology on a differentiable manifold M is Hausdorff or T_2 (given two distinct points of M there exist neighborhoods of these two points which do not intersect each oithe)r and such that M can be covered by a countable number of coordinate neighborhoods (or M has a countable basis).

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Theorem (Whitney)

Any differentiable manifold M of dimension n which is Hausdorff and has a countable basis can be immersed in $\mathbb{R}^{2n}$ and embedded in $\mathbb{R}^{2n+1}$.

Partition of unity

Definition

We say that a family $\{f_\alpha\}_\alpha$ of differentiable functions $f_\alpha : M \to \mathbb{R}$ is a *differentiable partition of unity* if

- for all α $f_\alpha \geq 0$ and $\text{supp} f_\alpha \subset \mathbf{x}_\alpha(U_\alpha)$, where $\{(U_\alpha, \mathbf{x}_\alpha)\}_\alpha$ is a differentiable structure of M;

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Theorem

A differentiable manifold M has a differentiable partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.