Some Results in Riemannian Geometry part 1

Fabio Vlacci

MIGe Università di Trieste

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Something more on the topology of a differentiable manifold

From now on we'll always assume that the topology on a differentiable manifold M is Hausdorff or T_2 (given two distinct points of M there exist neighborhoods of these two points which do not intersect each oithe)r and such that M can be covered by a countable number of coordinate neighborhoods (or M has a countable basis).

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Theorem (Whitney)

Any differentiable manifold M of dimension n which is Hausdorff and has a countable basis can be immersed in $\mathbb{R}^{2n}$ and embedded in $\mathbb{R}^{2n+1}$.

Partition of unity

Definition

We say that a family $\{f_\alpha\}_\alpha$ of differentiable functions $f_\alpha : M \to \mathbb{R}$ is a *differentiable partition of unity* if

- for all α $f_\alpha \geq 0$ and $\mathrm{supp} f_\alpha \subset \mathbf{x}_\alpha(U_\alpha)$, where $\{(U_\alpha, \mathbf{x}_\alpha)\}_\alpha$ is a differentiable structure of M;

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- $\sum_\alpha f_\alpha(p) = 1$ for any $p \in M$.

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- $\{\mathbf{x}_\alpha(U_\alpha)\}_\alpha$ is locally finite;
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Theorem

A differentiable manifold M has a differentiable partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.

Riemannian Metric

Definition

A *Riemannian metric* is a smooth assignment:

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

such that g_p is an inner product for every p, i.e g_p is a symmetric, bilinear and positive-definite form on T_pM for every $p \in M$.

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In local coordinates:

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle .$$

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$$(g_{ij})_{ij}$$

is called local representation of the Riemannian metric g.

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A differentiable manifold M equipped with a Riemannian metric g will be called a *Riemannian manifold*.

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Definition

Given two Riemannian manifolds, a differentiable mapping $F : M \to N$ is called a *local isometry* at $p \in M$ if there is a neighborhood U of p in M such that
1) $F|_U : U \to F(U)$ is a (local) diffeomorphism
2) $\forall v, w \in T_qM$ $(q \in U)$ it turns out that

$$\langle v, w \rangle_q = \langle dF_q(v), dF_q(w) \rangle_{F(q)}$$

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If $U = M$ then F is called an *isometry*.

Isometric immersion

Remark

Let M, N be two differentiable manifolds and assume $F : M \to N$ is an immersion; then if N is equipped with a Riemannian metric, for $\forall v, w \in T_pM$ $p \in M$ define

$$\langle v, w \rangle_p := \langle dF_p(v), dF_p(w) \rangle_{F(p)}.$$

Since dF_p is injective, the symmetric bilinear form $\langle \ \rangle_p$ is positive definite, hence it is a Riemmanian metric on M which is generally called the metric induced *by F. Furthermore, F with these Riemannian metric becomes an* isometric immersion.

Existence of Riemannian metrics

Theorem

Any differentiable manifold M which is Hausdorff and has countable basis has a Riemannian metric.

WARNING

Remark (on notation)

In many text books (including "Riemannian Geometry" by do Carmo) if M is a differentiable manifold, then

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$$\Gamma(TM) = \mathfrak{X}(M);$$

furthermore,

$$\frac{\partial}{\partial x_j} := X_j.$$

Finally $\mathcal{D}(M)$ will denote the ring of smooth real-valued functions defined on M.

Affine Connection

Definition

An *affine connection* on a differentiable manifold M is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

(denoted by $(X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \nabla_X Y$) which satisfies the following properties:

- $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for any $f, g \in \mathcal{D}(M)$ ($\mathcal{D}(M)$-linearity in the first variable);

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- $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$ (additivity in the second variable);
- $\nabla_X(fY) = f\nabla_X(Y) + X(f)Y$ (Leibniz rule for the second variable)

Affine Connection (in local coordinates)

If

$$X = \sum_j x_j \frac{\partial}{\partial x_j} = \sum_j x_j X_j \qquad Y = \sum_k y_k \frac{\partial}{\partial x_k} = \sum_k y_k X_k$$

then

$$\nabla_X Y = \nabla_{\sum_j x_j X_j} \left(\sum_j y_k X_k \right) = \sum_j x_j \nabla_{X_j} \left(\sum_j y_k X_k \right) =$$
$$= \sum_{j,k} x_j y_k \nabla_{X_j} X_k + \sum_{j,k} x_j X_j(y_k) X_k$$

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$$= \sum_{j,k} x_j y_k \nabla_{X_j} X_k + \sum_{j,k} x_j X_j(y_k) X_k$$

put $\nabla_{X_j} X_k := \sum_k \Gamma^i_{jk} X_i$ so that

$$\nabla_X Y = \sum_i \left(\sum_{j,k} x_j y_k \Gamma^i_{jk} + X_j(y_k) \right) X_i$$

Affine Connection - a geometric interpretation

Proposition

Let M be a differentiable manifold, $\alpha : I \to M$ a differentiable curve and ∇ an affine connection on M. If $Y \in \mathfrak{X}(M)$, consider $V(t) := Y(\alpha(t))$, i.e. the restriction of Y along α.

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$$\frac{DV}{dt} := \nabla_{\frac{d\alpha}{dt}} Y$$

is the covariant derivative of V along α, *that is to say, if W is the restriction of another vector field along α and $f \in \mathcal{D}(M)$, it turns out that*

$$\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$$

$$\frac{D(fV)}{dt} = \frac{df}{dt} V + f \frac{DV}{dt}$$

Proof.

Indeed, if, in a local parametrization, one writes

$$\frac{d\alpha}{dt} = d\alpha\left(\frac{d}{dt}\right) = \sum_j \frac{dx_j}{dt} X_j|_\alpha$$

and

$$Y = \sum_k y_k X_k \quad V = \sum_k v_k X_k|_\alpha \quad \text{with } v_k = y_k|_\alpha$$

then

$$\nabla_{\frac{d\alpha}{dt}} Y = \nabla_{\sum_j \frac{dx_j}{dt} X_j} Y = \sum_j \frac{dx_j}{dt} \nabla_{X_j} \sum_k y_k X_k .$$

Therefore

$$\frac{DV}{dt} = \sum_{j,k}\left(\frac{dx_j}{dt} X_j(v_k) X_k + \frac{dx_j}{dt} v_k \nabla_{X_j} X_k\right) = \sum_{j,k}\left(\frac{dv_k}{dt} X_k + \frac{dx_j}{dt} v_k \nabla_{X_j} X_k\right)$$

Proof.

If $W = \sum_k w_k X_k|_\alpha$, then $(V+W)|_\alpha = \sum_k (v_k + w_k) X_k|_\alpha$ and so

$$
\begin{aligned}
\frac{D(V+W)}{dt} &= \sum_{j,k} \left(\frac{d(v_k + w_k)}{dt} X_k + \frac{dx_j}{dt} (v_k + w_k) \nabla_{X_j} X_k \right) \\
&= \sum_{j,k} \left(\frac{dv_k}{dt} X_k + \frac{dx_j}{dt} v_k \nabla_{X_j} X_k \right) \\
&+ \sum_{j,k} \left(\frac{dw_k}{dt} X_k + \frac{dx_j}{dt} w_k \nabla_{X_j} X_k \right) \\
&= \frac{DV}{dt} + \frac{DW}{dt}
\end{aligned}
$$

□

Proof.

If $f \in \mathcal{D}(M)$, then $fV|\alpha = \sum_k fv_k X_k|_\alpha$ and so

$$\begin{aligned}
\frac{D(fV)}{dt} &= \sum_{j,k} \left(\frac{d(fv_k)}{dt} X_k + \frac{dx_j}{dt} fv_k \nabla_{X_j} X_k \right) \\
&= \sum_{j,k} \left(\frac{df}{dt} v_k X_k + f \frac{dv_k}{dt} X_k + \frac{dx_j}{dt} fv_k \nabla_{X_j} X_k \right) \\
&= \frac{df}{dt} V + f \frac{DV}{dt}
\end{aligned}$$

□

Definition

Let M be a differentiable manifold with an affine connection ∇ on M. A vector field V along a differentiable curve $\alpha : I \to M$ is called *parallel* if $\frac{DV}{dt} = 0$ for all $t \in I$.

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Proposition

Let M be a differentiable manifold with an affine connection ∇ on M. Consider a differentiable curve $\alpha : I \to M$ and a vector $V_0 \in T_{\alpha(t_0)}M$ with $t_0 \in I$. Then there exists a unique parallel vector field V along α such that $V(t_0) = V_0$.

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Definition

The vector field V in the previous Proposition is called the *parallel transport* of V_0 along α.

Proof.

Locally, V is the solution of

$$\begin{cases} \dfrac{DV}{dt} = 0 \\ V(t_0) = V_0; \end{cases}$$

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since

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and

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and

$$\nabla_{X_j} X_k = \sum_{\ell} \Gamma^{\ell}_{jk} X_{\ell}$$

$$\frac{DV}{dt} = 0 \iff \sum_{\ell} \left(\frac{dv_\ell}{dt} + \sum_{j,k} \frac{dx_j}{dt} v_k \Gamma^{\ell}_{jk} \right) X_\ell = 0$$

Proof.

In other words, since the vector fields $\{X_j\}_j$ are linearly independent, each of the components v_ℓ of V has to satisfy the Cauchy problem

$$\begin{cases} \dfrac{dv_\ell}{dt} + \sum_{j,k} \dfrac{dx_j}{dt} v_k \Gamma^\ell_{jk} & = & 0 \\ \\ v_\ell(t_0) & = & [V_0]_\ell \end{cases}$$

□

Geodesics

Definition

Let M be a differentiable manifold with an affine connection ∇ on M. A differentiable curve $\gamma : I \to M$ is called a *geodesic* at $t_0 \in I$ if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)|_{t=t_0} = \frac{D\gamma'}{dt}|_{t=t_0} = 0;$$

if $\gamma : I \to M$ is a *geodesic* at any $t_0 \in I$, then we say that γ is a geodesic (in M).

Geodesics

In local parametrization,

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$$

if and only if

$$\frac{d^2x_k}{dt^2} + \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0.$$

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Geodesic equation is a second-order ODE. By standard ODE theory, for every $(p, v) \in TM$, there exists a unique geodesic γ with $\gamma(0) = p$, $\gamma'(0) = v$

Geodesics

Indeed, any differentiable curve $\gamma : I \to M$ in M determines a curve in TM, namely $t \mapsto (\gamma(t), \gamma'(t))$. If γ is a geodesic, then, locally, the curve

$$t \mapsto (x_1(t), \ldots, x_n(t), \frac{dx_1(t)}{dt}, \ldots, \frac{dx_n(t)}{dt})$$

satisfies the system

$$\begin{cases} \frac{dx_k}{dt} & = & y_k \\ \frac{dy_k}{dt} & = & -\sum_{i,j} \Gamma_{ij}^k y_i y_j \end{cases}$$

for any $k \in \{1, \ldots, n\}$ and with initial data $(x_1(0), \ldots, x_n(0)) = p$ $(y_1(0), \ldots, y_n(0)) = v$.

Riemannian connections

Definition

Let M be a differentiable manifold with an affine connection ∇ on M and a Riemannian metric $\langle\ ,\ \rangle$. The connection ∇ is said to be *compatible* with the metric $\langle\ ,\ \rangle$ if any vector fields V and W along the differentiable curve $\alpha : I \to M$ we have

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$$\frac{d}{dt}\langle V, W\rangle = \langle\frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$$

In particular, if V, W are parallel vector fields along α, then $\langle V, W\rangle$ is constant along α.

Riemannian connections

Proposition

A connection ∇ on M is compatible with the Riemannian metric $\langle\ ,\ \rangle$ if and only if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_x Z\rangle$$

for any $X, Y, Z \in \mathcal{X}(M)$.

Remark

If $\gamma : I \to M$ is a geodesic in M equipped with an affine connection ∇ compatible with the metric, then

$$\frac{d}{dt}\langle\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle = 2\langle\frac{D}{dt}\left(\frac{d\gamma}{dt}\right),\frac{d\gamma}{dt}\rangle = 0$$

meaning that the length of the tangent vector $\dfrac{d\gamma}{dt}$ *is constant.*

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meaning that the length of the tangent vector $\frac{d\gamma}{dt}$ is constant. To avoid the possibility for a geodesic to be a point, we assume that $\left|\frac{d\gamma}{dt}\right| = k \neq 0$. Hence the arc length s of γ starting from t_0 is

$$s(t) = \int_{t_0}^{t} \left|\frac{d\gamma}{d\tau}\right| d\tau = k(t - t_0)$$

therefore the parameter t of a geodesic is affinely equivalent to the arc length of the geodesic.

Symmetric connections

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In local parametrization, if ∇ is symmetric

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0$$

which implies

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i$$

or, equivalently,

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

Symmetric connections

Definition

$$\nabla_X Y - \nabla_Y X - [X, Y] := T(X, Y)$$

is called the *torsion* of ∇.
If $T \equiv 0$ we say that ∇ is *torsion free* (or symmetric).

Levi-Civita Theorem

Theorem (Levi-Civita).

Given a Riemannian manifold M, there exists a unique connection ∇ symmetric and compatible with the Riemannian metric, i.e. such that, for any $X, Y, Z \in \mathcal{X}(M)$,

1. $\nabla_X Y - \nabla_Y X = [X, Y]$
2. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

Proof Sketch of Uniqueness

Starting from metric compatibility:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Write similar identities by cyclic permutation.
Adding and subtracting:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle$$
$$+\langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle.$$

Thus ∇ is uniquely determined by g.

In local coordinates:

$$\langle X_i, X_j \rangle = g_{ij}$$

and

$$\nabla_{X I_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

with

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(X_i g_{jl} + X_j g_{il} - X_l g_{ij} \right).$$