Lesson 9 - 12/3/2026

The Radon - Nikodym theorem

def. Let λ be a positive measure
Let ν be a signed or complex measure $\Big\}$ on $(\Omega, \mathcal{F})$

I say that ν is absolutely continuous w.r.t. λ
if $\forall A \in \mathcal{F}$, $\lambda(A) = 0 \Rightarrow \nu(A) = 0$
I write $\nu \ll \lambda$

theorem: let λ and ν as above
$\nu \ll \lambda \iff |\nu| \ll \lambda$

proof. if $|\nu| \ll \lambda$ then, since $|\nu(A)| \le |\nu|(A)$, $\nu \ll \lambda$
$\left(|\nu|(A) = \sup\left\{ \sum_{j=1}^{n} |\nu(A_j)| \text{ if } A = A_1 \cup \dots \cup A_n \text{ pairwise disjoint} \right\} \right)$

conversely
suppose $\nu \ll \lambda$

take $A \in \mathcal{F}$ s.t. $\lambda(A) = 0$
consider $A = A_1 \cup A_2 \cup \dots \cup A_n$ p.d.
λ is positive $\lambda(A) = 0 \Rightarrow \lambda(A_j) = 0$ for $j = 1, \dots, n$
then $\nu(A_j) = 0$ for $j = 1 \dots n$
then $\sum_{j=1}^{n} |\nu(A_j)| = 0$
then $|\nu|(A) = \sup\left\{ \sum_{j=1}^{n} |\nu(A_j)| \right\} = 0 \Rightarrow |\nu|(A) = 0$
QED

Theorem. Let ν be a complex measure $(\Rightarrow \forall A \in \mathcal{F}, |\nu(A)| < +\infty)$
Let λ be a positive measure
then
$$\nu \ll \lambda \iff \overbrace{\forall \varepsilon > 0, \exists \delta > 0 : \forall A \in \mathcal{F}, \ \lambda(A) < \delta \Rightarrow |\nu(A)| < \varepsilon}^{\circledast}$$

(rem. $\nu = \nu_f$ $f \in L^1_\lambda$
this is notion abs. cont. of the integral
$\nu_f(A) = \int_A f \, d\lambda$
$\lambda(A) < \delta \Rightarrow \left| \int_A f \, d\lambda \right| < \varepsilon$)

proof
suppose $\circledast$ is valid
let $A \in \mathcal{F}$ s.t. $\lambda(A) = 0$
the $\forall \varepsilon > 0$, $|\nu(A)| < \varepsilon$ then $|\nu(A)| = 0$
$\circledast \Rightarrow \nu \ll \lambda$

for showing the converse I show that
$\nu \ll \lambda \Rightarrow |\nu| \ll \lambda \underset{\substack{\uparrow \\ \text{I prove}}}{\Rightarrow} \forall \varepsilon > 0, \exists \delta > 0 : \forall A \in \mathcal{F}$ if $\lambda(A) < \delta$ then $|\nu|(A) < \varepsilon$
$\Downarrow$
$\circledast$ since $|\nu(A)| \le |\nu|(A)$

Let $|\nu| \ll \lambda$
suppose by contradiction
$\exists \varepsilon_0 > 0 : \forall \delta > 0 \ \exists A_\delta \in \mathcal{F}$ s.t. $\lambda(A_\delta) < \delta \wedge |\nu|(A_\delta) \ge \varepsilon_0 > 0$

I take $\delta = \frac{1}{2^n}$ $\exists A_n \in \mathcal{F}$ s.t. $\lambda(A_n) < \frac{1}{2^n} \wedge |\nu|(A_n) \ge \varepsilon_0 > 0$

I set $B_n = \bigcup_{k \ge n} A_n$ $\quad (B_n \supseteq B_{n+1} \ \forall n)$

$C = \bigcap_n B_n$

$\lambda(B_n) \le \sum_{k=n}^{+\infty} \lambda(A_n) = \sum_{k=n}^{+\infty} \frac{1}{2^n} = \frac{1}{2^{n-1}}$

$\lambda(C) = \lim_n \lambda(B_n) \le \lim_n \frac{1}{2^{n-1}} = 0$

but $|\nu| \ll \lambda \Rightarrow |\nu|(C) = 0$

but then $\lim_n |\nu|(B_n)$ must be $= 0$ $\quad$ (ν complex $\Downarrow$ $|\nu|$ positive finite $|\nu|(B_1) < +\infty$ so that $\lim |\nu|(B_n) = |\nu|(C)$)

but $B_n \supseteq A_n$
so $|\nu|(B_n) \ge |\nu|(A_n) \ge \varepsilon_0 > 0 \ \forall n$
$\Rightarrow \lim_n |\nu|(B_n)$ cannot be 0. QED

Remark. this property is valid also for signed measure provided that, for instance, the following is valid

$$\lambda(A) < +\infty \Rightarrow |\nu(A)| < +\infty.$$

Remark

let λ be a positive measure

¿ how to construct (easily) a signed or complex measure which is abs. cont. with respect to λ ?

take $f \in L^1_\lambda$ define $\nu_f(A) := \int_A f \, d\lambda$

Th. (Radon - Nikodym)

let λ be a positive measure, $\underline{\sigma\text{-finite}}$.

let ν be a complex or signed measure

suppose $\nu \ll \lambda$

then there exist f_0 measurable such that

$\forall A \in \mathcal{F}$ if $|\nu(A)| < +\infty$ then $f_0 \cdot \chi_A \in L^1_\lambda$

and $\nu(A) = \int_A f_0 \, d\lambda$

(if ν is complex the thesis is $\exists f_0 \in L^1_\lambda$ s.t. $\forall A \in \mathcal{F}$ $\nu(A) = \int_A f_0 \, d\lambda$)