Lesson 9 - 13/3/2026

Th (Radon - Nikodym)

Let λ be a positive, σ-finite measure
Let μ be a signed or complex measure } on $(\Omega, \mathcal{F})$

suppose $\mu \ll \lambda$

then there exists f_0 measurable, s.t.

$\forall A \in \mathcal{F}$, $|\mu(A)| < +\infty \Rightarrow f_0 \chi_A \in L^1_\lambda$ and $\mu(A) = \int_A f_0 \, d\lambda = \int_\Omega f_0 \chi_A \, d\lambda$

(remark that if μ is complex measure then the conclusion is $f_0 \in L^1_\lambda(\Omega)$ and $\forall A \in \mathcal{F}$, $\mu(A) = \int_A f_0 \, d\lambda$)

proof (only in the case λ, μ positive and finite measures)

Let $\mathcal{E} = \{ f \in L^1_\lambda(\Omega) : f \geq 0$ and $\forall A \in \mathcal{F}, \int_A f \, d\lambda \leq \mu(A) \}$

$\mathcal{E} \neq \emptyset$ since $0 = f \in \mathcal{E}$

denote by $\alpha = \sup_{f \in \mathcal{E}} \{ \int_\Omega f \, d\lambda \}$ $\quad (0 \leq \alpha \leq \mu(\Omega))$

consider $(f_n)_n$ in $\mathcal{E}$ s.t. $\int_\Omega f_n \, d\lambda \uparrow \alpha$

consider $g_n(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$

$g_n \geq 0$, $g_{n+1}(x) \geq g_n(x)$ $\forall n \forall x$ and $g_n \in \mathcal{E}$

(to see this $\int_A g_n \, d\lambda = \sum_{j=0}^{n} \int_{A_j} f_j(x) \, d\lambda$ where $A_j = \{x \in A$ s.t. $f_j(x) = \max\{f_1(x), f_2(x), \dots f_n(x)\} = g_n(x)\}$

since $f_j \in \mathcal{E}$, $\int_{A_j} f_j(x) \, d\lambda \leq \mu(A_j)$

$\leq \sum_{j=0}^{n} \mu(A_j) = \mu(A)$)

since $g_n \geq f_n$, $\lim_n \int_\Omega g_n \, d\lambda = \alpha$

finally I apply to (g_n) the Beppo Levi theorem

so $g = \lim_n g_n$ $\quad g \in L^1_\lambda(\Omega)$, $g \in \mathcal{E}$, $\int_\Omega g \, d\lambda = \alpha$

In particular $\int_A g \, d\lambda \leq \mu(A)$ $\forall A \in \mathcal{F}$

Now I define $\nu(A) = \mu(A) - \int_A g \, d\lambda$ $\quad \nu$ is positive measure

if $\nu \equiv 0$ the theorem ~~is~~ proved

$\Updownarrow$

$\nu(\Omega) = 0$

By contradiction suppose $\nu(\Omega) > 0$

I consider $k \in \mathbb{R}$ s.t. $\lambda(\Omega) - k\,\nu(\Omega) < 0$

so I define $\nu_1(A) = \lambda(A) - k\,\nu(A)$, ν_1 is a signed measure

Consider Hahn's decomposition for ν_1, (N, P)

now $\boxed{\lambda(N) > 0}$

in fact if $\lambda(N) = 0$ then $\mu(N) = 0$ $(\mu \ll \lambda)$

then $\underbrace{\mu(N)}_{=0} - \underbrace{\int_N g \, d\lambda}_{\lambda(N)=0,\ =0} = 0$ then $\nu(N)$

then $\nu_1(N) = \lambda(N) - k\nu(N) = 0$ impossible

define $h(x) = \begin{cases} \frac{1}{k} & \text{if } x \in N \\ 0 & \text{if } x \in P \end{cases}$

define $h(x) = \begin{cases} \frac{1}{k} & \text{if } x \in N \\ 0 & \text{if } x \in P \end{cases}$

evaluate

$$\int_A (g+h)\,d\lambda = \int_A g\,d\lambda + \int_A h\,d\lambda$$

$$= \int_A g\,d\lambda + \frac{1}{k}\lambda(A\cap N)$$

now

$$0 \geq \nu_k(A\cap N) = \lambda(A\cap N) - k\,\nu(A\cap N)$$

$$\Rightarrow \frac{1}{k}\lambda(A\cap N) \leq \nu(A\cap N) = \mu(A\cap N) - \int_{A\cap N} g\,d\lambda$$

$$\leq \int_A g\,d\lambda + \mu(A\cap N) - \int_{A\cap N} g\,d\lambda$$

$$\int_A (g+h)\,d\lambda \leq \int_{A\cap P} g\,d\lambda + \mu(A\cap N) \qquad \int_{A\cap P} g\,d\lambda \leq \mu(A\cap P)$$

$$\int_A (g+h)\,d\lambda \leq \mu(A\cap P) + \mu(A\cap N) = \mu(A)$$

conclusion $\boxed{g+h \in \mathcal{C}}$

$$\int_\Omega (g+h)\,d\lambda = \underbrace{\int_\Omega g\,d\lambda}_{=\alpha} + \underbrace{\int_\Omega h\,d\lambda}_{=\frac{1}{k}\lambda(N) > 0} > \alpha \quad \text{impossible!}$$

$$\alpha = \sup_{f\in\mathcal{C}} \int_\Omega f\,d\lambda$$

QED

Rem. the hypothesis of λ (positive) σ-finite cannot be avoided

I show a counter example.

Let $\Omega = \mathbb{R}^d$, $\mathcal{A} = \mathcal{B} = \{\text{borelian sets}\}$

let $\lambda : \mathcal{B} \to [0,+\infty]$ measure which counts the points

let μ be the Lebesgue measure

λ is not σ-finite

obviously $\mu \ll \lambda$

suppose by contr. that $\mu(A) = \int_A f_0\,d\lambda$

consider $A = \{x_0\}$

$$0 = \mu(\{x_0\}) = \int_{\{x_0\}} f_0(x)\,d\lambda = f_0(x_0) \Rightarrow f_0(x_0) = 0 \quad \forall x_0 \in \mathbb{R}^d$$

$$\Rightarrow f_0 \equiv 0$$

$$\Rightarrow \forall A \in \mathcal{B},\ \mu(A) = 0$$

Lebesgue, impossible

Corollary

let μ and λ as in R-N theorem

let $\mu \ll \lambda$

then there exists f_0 measurable such that

$\forall f \in L^1_\mu(\Omega)$, $f\cdot f_0 \in L^1_\lambda(\Omega)$

and $\int_\Omega f\,d\mu = \int_\Omega f f_0\,d\lambda$

Remark Let $F : \mathbb{R} \to \mathbb{R}$ F increasing and continuous from the right

it is possible to prove that there exists a unique measure μ_F on Borel set of $\mathbb{R}$

$$\mu_F(]a,b]) = F(b) - F(a)$$

μ_F is called the Lebesgue - Stieltjes measure associated to F

$$\underset{\substack{\uparrow \\ \text{Lebesgue measure}}}{\mu_F << \lambda} \iff F_{|[a,b]} \in AC([a,b]) \text{ for all } [a,b]$$

Remark

consider $F : \mathbb{R} \to [0,1]$

F increasing, F continuous from the right,

$\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

(F is the distribution function of a random variable)
funzione di ripartizione

I remark that F has at most a countable set of points in which it is not continuous

I define $j_F : \mathbb{R} \to \mathbb{R}$, $j_F(t) = F(t) - \lim_{x \to t^-} F(t)$

I define $\chi(t) = \sum_{x_n \leq t} j_F(x_n)$ $x_n \in \{x \in \mathbb{R} : j_F(x) \neq 0\}$

$F_1 = F - \chi$

F_1 is continuous, increasing,

consider $F_1' = f$

consider $F_1(t) - \int_{-\infty}^{t} f(s)\,ds = G(t)$

(show this)

$G(t)$ is continuous, increasing, and $G'(t) = 0$ a.e.

conclusion

$$F(t) = \chi(t) + G(t) + \underbrace{\int_{-\infty}^{t} F_1'(s)\,ds}_{F_2(t)}$$

discrete distribution

singular part,

absolutely continuous dist.

The Hardy - Littlewood maximal function.

Let $\mathcal{B}$ be the σ- algebra of Borel's set of $\mathbb{R}^d$

Let ν be a complex measure on $\mathcal{B}$

I denote by λ the ~~L~~ Lebesgue measure.

def. if $\displaystyle\lim_{r\to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))}$ exists

this limit is the symmetric derivative of ν w.r.t. λ at x

I denote it as $\frac{d\nu}{d\lambda}(x)$

Rem. let $\nu = \nu_f$ for $f \in L^1(\mathbb{R}^d)$ $\quad \nu_f(B(x,r)) = \int_{B(x,r)} f(y)\,dy$

$$\frac{d\nu_f}{d\lambda}(x) = \lim_{r\to 0^+} \underbrace{\frac{1}{\lambda(B(x,r))} \cdot \int_{B(x,r)} f(y)\,dy}_{\text{integral mean}}$$

we are interested in the existence of $\frac{d\nu}{d\lambda}$ and in its value

def let ν a complex measure on $\mathcal{B}$.

I define $\quad M_\nu(x) = \displaystyle\sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}$

$\uparrow$

Hardy - Littlewood maximal function

$M_\nu : \mathbb{R}^d \to [0,+\infty]$

Lemma M_ν is lower semicontinuous function

i.e. $\forall \alpha \geq 0$, $\{x\in\mathbb{R}^d : M_\nu(x) > \alpha\}$ is an open set

proof. let $E = \{x \in \mathbb{R}^d : M_\nu(x) > \alpha\}$

$M_\nu(x) = \displaystyle\sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}$

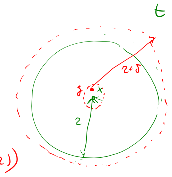
$\sup \dots > \alpha$

$\exists r>0$ s.t. $\dfrac{|\nu|(B(x,r))}{\lambda(B(x,r))} > t' > t > \alpha \quad (\exists\, t', t)$

I consider $\delta > 0$ s.t. $\dfrac{(r+\delta)^d}{r^d} < \dfrac{t'}{t}$ $\quad$ (δ exists! because $\frac{t'}{t} > 1$)

so that $\quad B(y, r+\delta) \supseteq B(x,\delta)$

take $|x-y| < \delta$



so $\quad |\nu|(B(y,r+\delta)) \geq |\nu|(B(x,r))$

and $\quad \lambda(B(y,r+\delta)) = \dfrac{(r+\delta)^d}{r^d} \cdot \lambda(B(x,\delta))$

$$\frac{|\nu|(B(y,r+\delta))}{\lambda(B(y,r+\delta))} \geq \frac{|\nu|(B(x,r))}{\frac{(r+\delta)^d}{r^d}\lambda(B(x,r))} = \frac{r^d}{(r+\delta)^d} \underbrace{\frac{|\nu|(B(x,r))}{\lambda(B(x,r))}}_{> t'}$$

consequently

$$\frac{|\nu|(B(y,r+\delta))}{\lambda(B(y,r+\delta))} > \frac{r^d}{(r+\delta)^d} \cdot t'$$

but $\dfrac{(r+\delta)^d}{r^d} < \dfrac{t'}{t}$

$\dfrac{r^d}{(r+\delta)^d} > \dfrac{t}{t'}$

$$> \frac{t}{\cancel{t'}} \cdot \cancel{t'} = t > \alpha$$

Conclusion if $|x-y| < \delta$ then $M_\nu(y) > \alpha$

$\Rightarrow$ E is open

Ex try to prove Wiener lemma

Lemma (Wiener)

consider $B(x_1, r_1), B(x_2, r_2), B(x_3, r_3), \dots, B(x_N, r_N)$

N balls in $\mathbb{R}^d$

then $\exists$ $S \subseteq \{1, 2, \dots N\}$

s.t. for $i, j \in S$, $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$

and $\displaystyle\bigcup_{i=1}^{N} B(x_i, r_i) \subseteq \bigcup_{j \in S} B(x_j, 3r_j)$